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Dr. Zhang Wenpeng
Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R.China

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Contents

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Amenable partial orders on locally inverse semigroup [1]

Bin Li

Department of Project Management,
Shaanxi Business College, Xi'an, Shaanxi, P. R. China
Email: libinsxdd@126.com

Abstract Suppose that S is a locally inverse semigroup with an inverse transversal S°. In this paper, we first introduce the CS°-cone of S, which modify the definitions of cone in [3] and S°-cone in [4]. By an CS°-cone, we can construct an amenable partial order on S. Conversely, every amenable partial order on S can be constructed in this way. It is easily verified that the set $E(S^{\circ})$ of all idempotent elements of S° is the smallest CS°-cone of S. Also, we show that the amenable partial order constructed by $E(S^{\circ})$ is equal to the natural partial order on S and so the natural partial order on S is the smallest amenable partial order.

Keywords Locally inverse semigroup, amenable partial order, inverse transversal, order -preserving bijection.

2000 Mathematics Subject Classification: 20M10, 06B10

§1. Introduction and preliminaries

A semigroup S is said to be a partially ordered semigroup, or to be partially ordered, if it admits a compatible ordering $\leq$; that is, $\leq$ is a partial order on S such that $(\forall\, a,\, b \in S,\, x \in S^1)$ $a \leq b \Longrightarrow xa \leq xb$ and $ax \leq bx$.

Let S be a regular semigroup with set $E(S)$ of idempotent elements. As usual, $\preceq$ denotes the natural partial order on S. That is, for any $a,\, b \in S$,

$$a \preceq b \quad \text{if and only if} \quad a = eb = bf \quad \text{for some } e,\, f \in E(S).$$

By corollary 4.2 in [1], the natural partial order $\preceq$ on S is compatible with the multiplication if and only if S is a locally inverse semigroup. Thus a locally inverse semigroup equipped with the natural partial order is a partially ordered semigroup. Particularly, an inverse semigroup is a partially ordered semigroup under the natural partial order.

Blyth, McFadden, McAlister and Almeida Santos introduced and studied amenable partially ordered inverse semigroup in [3, 5, 6, 7].

Definition 1.1.[3] Let $(S,\, \cdot,\, \leq)$ be a partially ordered inverse semigroup. The partial order $\leq$ is said to be a left(right) amenable partial order if it coincides with $\preceq$ on idempotents and

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for each $a,\, b \in S$, $a \leq b$ implies $a^{-1}a \preceq b^{-1}b$ $(aa^{-1} \preceq bb^{-1})$. If $\leq$ is both a left amenable partial order and a right amenable partial order on S, then $\leq$ is called an amenable partial order and S is called an amenable partially ordered inverse semigroup.

It is easy to see that the natural partial order on inverse semigroup is an amenable partial order. Suppose that S is an inverse semigroup. Also, McAlister [3] introduced the notion of a cone as the full subsemigroup Q of the subsemigroup $R(E(S)) = \{x \in S|\ (\forall e \in E(S)),\, exe = xe\}$ with the properties that $Q \bigcap Q^{-1} = E(S)$ and $xQx^{-1} \subseteq Q$ for all $x \in S$. He showed that there exists an order-preserving bijection between the set of cones in S and the set of left amenable partial orders on S.

Blyth and Almeida Santos generalized (left) amenable partial orders on inverse semigroup to regular semigroup with an inverse transversal in [4]. Let S be a regular semigroup, for any $a \in S$, $V(a)$ denotes the all inverses of a. An inverse, transversal of a regular semigroup S is an inverse subsemigroup S° with the property that $|S^\circ \bigcap V(a)| = 1$ for every a in S. The unique inverse of a in $S^\circ \bigcap V(a)$ is written as a° and $(a^\circ)^\circ$ as $a^{\circ\circ}$. The set of idempotents in S° is denoted by $E(S^\circ)$. We recall the following definition.

Definition 1.2. [4] Let $(S, \cdot, \leq)$ be a partially ordered regular semigroup with an inverse transversal S°. If $\leq$ coincides with $\preceq$ on idempotents and the partial order $\leq$ has the following property

$$(\forall a,\, b \in S)\ \ a \leq b \Longrightarrow a^\circ a \preceq b^\circ b,$$

then $\leq$ is said to be a left amenable partial order on S. Dually, if $a \leq b$ implies $aa^\circ \preceq bb^\circ$, then $\leq$ is called a right amenable partial order on S. If $\leq$ is both a left amenable partial order and a right amenable partial order on S, then $\leq$ is called an amenable partial order and S is called an amenable partially ordered regular semigroup with inverse transversal S°.

A right normal orthodox semigroup is an orthodox semigroup in which the set $E(S)$ of all idempotent elements of S forms a right normal band, that is to say

$$(\forall e,\, f,\, g \in E(S))\ efg = feg.$$

Suppose that S is a right normal orthodox semigroup with inverse transversal S°. It is easy to see that the subset

$$R(E(S)) = \{a \in S|(\forall e \in E(S))\, eae = ae\}$$

of S is a subsemigroup of S.

Blyth and Almeida Snatos introduced the concept of S°-cone on S in [4], which extended the notion of cone in inverse semigroup. Assume that S is a right normal orthodox semigroup with an inverse transversal S°. A non-empty subset Q of S is called a S°-cone of S if Q satisfies the following three conditions:

(C1) Q is a subsemigroup of $R(E(S))$;

(C2) $Q \bigcap Q^\circ = E(S^\circ)$;

(C3) $(\forall x \in S)\ xQx^\circ \subseteq Q$.

Particularly, if Q is a S°-cone and $E(S) \subseteq Q$, then Q is said to be a locally maximal S°-cone of S.

Let S is a right normal orthodox semigroup with an inverse transversal S°, Blyth and Almeida Snatos in [4] proved that there is an order-preserving bijection from the set of all locally maximal S°-cones to the set of all left amenable orders definable on S and the natural partial order is the smallest left amenable partial order(see theorems 7 and 11 in [4]).

Suppose that S is a locally inverse semigroup with an inverse transversal S°. Blyth and Almeida Santos gave a complete description of all amenable partial orders on S and showed the natural partial order on S is the smallest amenable partial order in [5]. They also proved that every amenable partial orders on S° extends to a unique amenable partial order on S.

In this paper, we will give a new characterization of the amenable partial orders on S. We first introduce the CS°-cone of S, which modify the definitions of cone in [3] and S°-cone in [4]. Being similar to [5], by an CS°-cone, we can construct an amenable partial order on S. Conversely, every amenable partial order on S can be constructed in this way. It is easily verified that the set $E(S^\circ)$ of all idempotent elements of S° is the smallest CS°-cone of S. Also, we show that the amenable partial order constructed by $E(S^\circ)$ is equal to the natural partial order on S and so the natural partial order on S is the smallest amenable partial order. Finally, it is established that for any (but fixed) the inverse transversal S° of S, there is an order-preserving bijection from the set of all amenable partial orders on S to the set of all amenable partial orders on S° and so every amenable partial order on S° is uniquely extended to an amenable partial order on S.

§2. Constructing amenable partial orders

Suppose that $(S, \cdot)$ is a regular semigroup with an inverse transversal S°. The following two statements are needed. Blyth and Almeida Santos say in [5] that S satisfies the following formular

$$(\forall a, b \in S)\, (ab)^\circ = (a^\circ ab)a^\circ = b^\circ(ab^\circ b)^\circ = b^\circ(a^\circ abb^\circ)^\circ,\ (a^\circ b)^\circ = b^\circ a^{\circ\circ},\ (ab^\circ)^\circ = b^{\circ\circ}a^\circ. \quad (1)$$

According to Blyth and Almeida Santos in [5], if S is locally inverse, then

$$(\forall a, b, c \in S) \quad a^\circ bc^\circ = a^\circ b^{\circ\circ}c^\circ. \quad (2)$$

Suppose that S is a regular semigroup with an inverse transversal S°. Blyth and Almeida Santos say in ([4] and [5]) that the two subsets of $E(S)$

$$\Lambda = \{x^\circ x |\, x \in S\},\ I = \{xx^\circ |\, x \in S\}$$

are respectively right regular subband and left regular subband of $E(S)$. Hence, we immediately have

Lemma 2.1. Let S be a locally inverse semigroup with an inverse transversal S°. Then Λ is a right normal subband of $E(S)$ and I is a left normal subband of $E(S)$.

Proof. Suppose that S is a locally inverse semigroup with an inverse transversal S°. Consider the set

$$E(S^\circ)\zeta = \{x \in S^\circ | (\forall e \in E(S^\circ))\ ex = xe\},$$

which is the centralizer of $E(S^\circ)$ in S°(see section 5.3 in [2]). It is easy to see that $E(S^\circ)\zeta$ is a subsemigroup of S°. Now, we have

Definition 2.1. [5] Suppose that S is a locally inverse semigroup with an inverse transversal S°. A subset Q of S° will be called an CS°-cone if

(i) Q is a subsemigroup of $E(S^\circ)\zeta$;

(ii) $Q \bigcap Q^\circ = E(S^\circ)$;

(iii) $(\forall x \in S)\ x^\circ Q x^{\circ\circ} \subseteq Q$.

It is easy to see that $E(S^\circ)$ is an CS°-cone. The following result will show that an amenable partial order on S also can be constructed by an CS°-cone.

Theorem 2.1. Suppose that S is a locally inverse semigroup with an inverse transversal S°. Let C be an CS°-cone. Then the relation $\leq_C$ defined on S by

$$x \leq_C y \Longleftrightarrow xx^\circ \preceq yy^\circ,\ x^\circ x \preceq y^\circ y,\ x^\circ y^{\circ\circ},\ y^{\circ\circ}x^\circ \in C$$

is an amenable partial order on S.

Proof. It is easily seen that $\leq_C$ is reflexive. Suppose that $x \leq_C y$ and $y \leq_C x$. From the definition of $\leq_C$ we have $xx^\circ = yy^\circ$, $x^\circ x = y^\circ y$, $x^\circ y^{\circ\circ}$, $y^\circ x^{\circ\circ} \in C$. Thus $y^\circ x^{\circ\circ} = (x^\circ y^{\circ\circ})^\circ \in C \bigcap C^\circ = E(S^\circ)$, since C is an CS°-cone. It follows from $xx^\circ = yy^\circ$ and (1) that $x^{\circ\circ}x^\circ = (xx^\circ)^\circ = (yy^\circ)^\circ = y^{\circ\circ}y^\circ$ and so $y^\circ = y^\circ y^{\circ\circ} y^\circ = y^\circ x^{\circ\circ} x^\circ$, which gives $y^\circ \preceq x^\circ$. Likewise, $x^\circ \preceq y^\circ$ and so $x^\circ = y^\circ$, furthermore, we have $x^{\circ\circ} = y^{\circ\circ}$. Hence, $x = xx^\circ \cdot x^{\circ\circ} \cdot x^\circ x = yy^\circ \cdot y^{\circ\circ} \cdot y^\circ y = y$, this shows that $\leq_C$ is anti-symmetric. If $x \leq_C y$ and $y \leq_C z$, then $x^\circ x \preceq y^\circ y \preceq z^\circ z$, $xx^\circ \preceq yy^\circ \preceq zz^\circ$ and $x^\circ y^{\circ\circ}$, $y^\circ z^{\circ\circ} \in C$. We obtain from $xx^\circ \preceq yy^\circ$ that $xx^\circ yy^\circ = xx^\circ$. It follows from definition 2.1 that $x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} \in C$. Thus, we have

$$\begin{aligned} x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} &= x^\circ x x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} \\ &= x^\circ (xx^\circ yy^\circ) z^{\circ\circ} \\ &= x^\circ z^{\circ\circ}. \end{aligned}$$

Consequently $x^\circ z^{\circ\circ} \in C$, similarly, we have $z^{\circ\circ}x^\circ \in C$ and so $x \leq_C z$. Hence, $\leq_C$ is transitive. Thus we show that $\leq_C$ is a partial order on S.

Suppose that $x \leq_C y$. For any $z \in S$, we have

$$\begin{aligned} (zx)^\circ (zy)^{\circ\circ} &= x^\circ (zxx^\circ)^\circ (zy)^{\circ\circ} \\ &= x^\circ (zyy^\circ xx^\circ)^\circ (zy)^{\circ\circ} \\ &= x^\circ x^{\circ\circ} x^\circ y^{\circ\circ} (zy)^\circ (zy)^{\circ\circ} \\ &= x^\circ y^{\circ\circ} (zy)^\circ (zy)^{\circ\circ} \\ &\in CE(S^\circ) \\ &\subseteq C. \qquad\qquad (E(S^\circ) \subseteq C) \end{aligned}$$

and

$$\begin{aligned}(zy)^{\circ\circ}(zx)^{\circ} &= (zy)^{\circ\circ}x^{\circ}(zxx^{\circ})^{\circ} && \text{(by (1))}\\ &= (zyy^{\circ}y)^{\circ\circ}x^{\circ}(zxx^{\circ}yy^{\circ})^{\circ}\\ &= (zyy^{\circ})^{\circ\circ}y^{\circ\circ}x^{\circ}(zyy^{\circ}xx^{\circ})^{\circ}\\ &= (zyy^{\circ})^{\circ\circ}y^{\circ\circ}x^{\circ}x^{\circ\circ}x^{\circ}(zyy^{\circ})^{\circ}\\ &= (zyy^{\circ})^{\circ\circ}y^{\circ\circ}x^{\circ}(zyy^{\circ})^{\circ}\\ &\in C. && (y^{\circ\circ}x^{\circ}\in C)\end{aligned}$$

Hence, we have $(zx)^{\circ}(zy)^{\circ\circ}$, $(zy)^{\circ\circ}(zx)^{\circ}\in C$. It follows from theorem 8 in [5] that $zx(zx)^{\circ}\preceq zy(zy)^{\circ}$ and $(zx)^{\circ}zx\preceq (zy)^{\circ}zy$. Thus, we see $zx\leq_C zy$, therefore $\leq_C$ is compatible on the left. Dually, we have that $\leq_C$ is also compatible on the right and so $(S,\cdot,\leq_C)$ is a partially ordered semigroup.

In the following, we will show that the partial order $\leq_C$ coincides with the natural partial order on $E(S)$. Suppose that $e, f\in E(S)$ and $e\leq_C f$. Then $e^{\circ}e\preceq f^{\circ}f$, $ee^{\circ}\preceq ff^{\circ}$ and so $e^{\circ}ef^{\circ}f=e^{\circ}e$, $ff^{\circ}ee^{\circ}=ee^{\circ}$. Pre-multiplying $e^{\circ}ef^{\circ}f=e^{\circ}e$ by e, we obtain $e=(ef^{\circ})f$, post-multiplying this by f, we have $ef=(ef^{\circ})f=e$. Similarly, we have $f(f^{\circ}e)=fe=e$, consequently $e\preceq f$. If $e, f\in E(S)$ and $e\preceq f$, then $e=ef=fe$, further, we have $e^{\circ}e(f^{\circ}f)=e^{\circ}ef(f^{\circ}f)=e^{\circ}ef=e^{\circ}e$, post-multiplying this by $e^{\circ}e$, we have $e^{\circ}e(f^{\circ}f)e^{\circ}e=e^{\circ}e$, by lemma 2.1, we have $f^{\circ}fe^{\circ}e=e^{\circ}e$, hence, $e^{\circ}e\preceq f^{\circ}f$, similarly, we have $ee^{\circ}\preceq ff^{\circ}$. It is clear that $e^{\circ}f^{\circ\circ}$, $f^{\circ\circ}e^{\circ}\in C$, hence, $e\leq_C f$. Consequently, $\leq_C$ coincides with $\preceq$ on idempotents. By definition 1.1, we have that $\leq_C$ is an amenable partial order.

Assume that S is a locally inverse semigroup with an inverse transversal S° and $\leq$ is a partial order on S. We denote by $\leq^{S^{\circ}}$ the restriction of $\leq$ on S°. Then the following lemma is clear.

Lemma 2.2. Suppose that S is a locally inverse semigroup with an inverse transversal S°. If $\leq$ is an amenable partial order on S, then $\leq^{S^{\circ}}$ is an amenable partial order on S°.

Lemma 2.3. Suppose that S is a locally inverse semigroup with an inverse transversal S°. If the partial order $\leq$ is an amenable partial order on S, then

$$(\forall a, b\in S)\ a\leq b\Longrightarrow a^{\circ\circ}\leq^{S^{\circ}} b^{\circ\circ}.$$

Proof. Suppose that $a\leq b$, then $aa^{\circ}\preceq bb^{\circ}$ and $a^{\circ}a\preceq b^{\circ}b$ by definition 1.1. Hence, we have $aa^{\circ}bb^{\circ}=aa^{\circ}$ and $a^{\circ}ab^{\circ}b=a^{\circ}a$, by (1) and (2), we have $(aa^{\circ}bb^{\circ})^{\circ}=(aa^{\circ}b^{\circ\circ}b^{\circ})^{\circ}=b^{\circ\circ}b^{\circ}a^{\circ\circ}a^{\circ}=(aa^{\circ})^{\circ}=a^{\circ\circ}a^{\circ}$. This shows that $a^{\circ\circ}a^{\circ}\preceq b^{\circ\circ}b^{\circ}$. Likewise, $a^{\circ}a^{\circ\circ}\preceq b^{\circ}b^{\circ\circ}$. Since $\leq$ is an amenable partial order, we have $a^{\circ\circ}a^{\circ}\leq b^{\circ\circ}b^{\circ}$ and $a^{\circ}a^{\circ\circ}\leq b^{\circ}b^{\circ\circ}$, consequently $a^{\circ\circ}=a^{\circ\circ}a^{\circ}aa^{\circ}a^{\circ\circ}\leq b^{\circ\circ}b^{\circ}bb^{\circ}b^{\circ\circ}=b^{\circ\circ}$. From $a^{\circ\circ}, b^{\circ\circ}\in S^{\circ}$ we have $a^{\circ\circ}\leq^{S^{\circ}} b^{\circ\circ}$, as required.

Proposition 2.1. Suppose that S is a locally inverse semigroup with an inverse transversal S°. If the partial order $\leq$ is an amenable partial order on S, then there exists an CS°-cone C such that $\leq_C=\leq$.

Proof. Assume that $\leq$ is an amenable partial order on S, we denote by $\leq^{S^{\circ}}$ the restriction of $\leq$ on S°. By lemma 2.3, we have $\leq^{S^{\circ}}$ is an amenable partial order on S°. Let

$$C=\{x|x\in S^{\circ},\ x^{\circ}x\leq^{S^{\circ}} x,\ xx^{\circ}\leq^{S^{\circ}} x\},$$

it is easily to see that $E(S^\circ) \subseteq C$. By lemma 2.1(iii) in [3] and its dual, we have C is the subset of $E(S^\circ)\zeta$. Now let $x,\, y \in C$. Then

$$\begin{aligned} (xy)^\circ xy &= y^\circ x^\circ xy && (x,\, y \in S^\circ) \\ &= y^\circ yy^\circ x^\circ xy \\ &\leq^{S^\circ} yy^\circ x^\circ xy \\ &= x^\circ xyy^\circ y && (E(S^\circ) \text{ is a semilattice}) \\ &= x^\circ xy \\ &\leq^{S^\circ} xy. \end{aligned}$$

Similarly, we have $xy(xy)^\circ \leq^{S^\circ} xy$. Hence, $xy \in C$. This shows that C is a subsemigroup of $E(S^\circ)\zeta$.

Suppose that $x, x^\circ \in C$. Then $x^\circ x \leq^{S^\circ} x$, $x^{\circ\circ}x^\circ \leq^{S^\circ} x^\circ$. From $x \in S^\circ$ we obtain $x^{\circ\circ} = x$, thus we have $xx^\circ \leq^{S^\circ} x^\circ$, post-multiplying this by x, we have $x \leq^{S^\circ} x^\circ x$ whence $x = x^\circ x \in E(S^\circ)$, hence, we have $C \bigcap C^\circ \subseteq E(S^\circ)$. On the other hand, it is clear that $E(S^\circ) \subseteq C \bigcap C^\circ$. Consequently $E(S^\circ) = C \bigcap C^\circ$.

For any $x \in S$, $a \in C$, we have

$$\begin{aligned} (x^\circ ax^{\circ\circ})^\circ(x^\circ ax^{\circ\circ}) &= x^\circ a^\circ x^{\circ\circ}x^\circ ax^{\circ\circ} && (\text{by } (1)) \\ &= x^\circ a^\circ x^{\circ\circ}x^\circ ax^{\circ\circ}x^\circ \cdot x^{\circ\circ} \\ &= x^\circ a^\circ ax^{\circ\circ}x^\circ \cdot x^{\circ\circ} && (a \in C \subseteq E(S^\circ)\zeta) \\ &= x^\circ a^\circ ax^{\circ\circ} \\ &\leq^{S^\circ} x^\circ ax^{\circ\circ}. && (a^\circ a \leq^{S^\circ} a) \end{aligned}$$

Dually, we have $(x^\circ ax^{\circ\circ})(x^\circ ax^{\circ\circ})^\circ \leq^{S^\circ} x^\circ ax^{\circ\circ}$. Thus we have $x^\circ ax^{\circ\circ} \in C$ and so $x^\circ Cx^{\circ\circ} \subseteq C$. It follows from definition 2.1 that C is an CS°-cone.

Consider the corresponding partial order $\leq_C$ given by $x \leq_C y \iff xx^\circ \preceq yy^\circ$, $x^\circ x \preceq y^\circ y, x^\circ y^{\circ\circ}$, $y^{\circ\circ}x^\circ \in C$. We can obtain from theorem 2.1 that $\leq_C$ is an amenable partial order on S. In the following, we will show that $\leq_C = \leq$.

Suppose that $x \leq_C y$. Then $xx^\circ \preceq yy^\circ$ and $x^\circ y^{\circ\circ} \in C$, hence, we have

$$\begin{aligned} x^{\circ\circ}x^\circ &= (xx^\circ yy^\circ)^{\circ\circ} \\ &= (xx^\circ y^{\circ\circ}y^\circ)^{\circ\circ} && (\text{by } (2)) \\ &= x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ && (\text{by } (1)) \\ &= y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ \\ &= y^{\circ\circ}(x^\circ y^{\circ\circ})^\circ(x^\circ y^{\circ\circ})y^\circ \\ &\leq^{S^\circ} y^{\circ\circ}(x^\circ y^{\circ\circ})y^\circ \\ &= y^{\circ\circ}x^\circ y^{\circ\circ}y^\circ \\ &= y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ \\ &= y^{\circ\circ}x^\circ(x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ) \\ &= y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ \\ &= y^{\circ\circ}x^\circ. \end{aligned}$$

Since $\leq$ is an amenable partial order, we have $x = xx^\circ x^{\circ\circ} x^\circ x \leq xx^\circ y^{\circ\circ} x^\circ x \leq yy^\circ y^{\circ\circ} y^\circ y = y$. Consequently $\leq_C \subseteq \leq$.

Suppose that $a, b \in S$ and $a \leq b$. It follows from $a \leq b$ that $aa^\circ \preceq bb^\circ$ and $a^\circ a \preceq b^\circ b$, furthermore, we have $b^\circ b^{\circ\circ} a^\circ = a^\circ$. By lemma 2.3, we have $a^{\circ\circ} \leq^{S^\circ} b^{\circ\circ}$. Hence, $(a^\circ b)^\circ (a^\circ b)^{\circ\circ} = b^\circ a^{\circ\circ} a^\circ b^{\circ\circ} \leq^{S^\circ} b^\circ b^{\circ\circ} a^\circ b^{\circ\circ} = a^\circ b^{\circ\circ} = (a^\circ b)^{\circ\circ}$, i.e., $(a^\circ b)^\circ (a^\circ b)^{\circ\circ} \leq^{S^\circ} (a^\circ b)^{\circ\circ}$. Similarly, we have $(a^\circ b)^{\circ\circ} (a^\circ b)^\circ \leq^{S^\circ} (a^\circ b)^{\circ\circ}$. Thus we obtain that $a^\circ b^{\circ\circ} = (a^\circ b)^{\circ\circ} \in C$. Similarly, we have $b^{\circ\circ} a^\circ \in C$. From the definition of $\leq_C$ we have $a \leq_C b$ and so $\leq \subseteq \leq_C$. Therefore $\leq_C = \leq$.

It is to see $E(S^\circ)$ is the smallest CS°-cone, by proposition 2.1 and theorem 11 in [5], we have

Theorem 2.2. Suppose that S is a locally inverse semigroup with an inverse transversal S°. Then $\leq_{E(S^\circ)}$ defined by $(*)$ is the smallest amenable partial order on S and $\leq_{E(S^\circ)} = \preceq$.

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Scientia Magna
Vol. 9 (2013), No. 3, 8-12

On some characterization of Smarandache -boolean near-ring with sub-direct sum structure

Dr. N. Kannappa † and Mrs. P. Tamilvani ‡

† TBML College, Porayar - 609307, Tamilnadu, India
E-mails: sivaguru91@yahoo.com

Abstract In this paper, we introduced Samarandache-2-algebraic structure of Boolean-near-ring namely Smarandache-Boolean-near-ring. A Samarandache-2-algebraic structure on a set N means a weak algebraic structure S_1 on N such that there exists a proper subset M of N, which is embedded with a stronger algebraic structure S_2, stronger algebraic structure means satisfying more axioms, that is $S_1 << S_2$, by proper subset one understands a subset different from the empty set, from the unit element if any, from the whole set [3]. We define Smarandache-Boolean-near-ring and obtain the some of its characterization through Boolean-ring with sub-direct sum structure. For basic concept of near-ring we refer to G. Pilz [11].

Keywords Boolean-ring, Boolean-near-ring, Smarandache-Boolean-near-ring, compatibility, maximal set, idempotent and uni-element.

§1. Preliminaries

Definition 1.1. A (Left) near ring A is a system with two Binary operations, addition and multiplication, such that

(i) the elements of A form a group $(A, +)$ under addition,

(ii) the elements of A form a multiplicative semi-group,

(iii) $x(y + z) = xy + xz$, for all x, y and $z \in A$. In particular, if A contains a multiplicative semi-group S whose elements generates $(A, +)$ and satisfy,

(iv) $(x + y)s = xs + ys$, for all $x, y \in A$ and $s \in S$, then we say that A is a distributively generated near-ring.

Definition 1.2. A near-ring $(B, +, \cdot)$ is Boolean-near-ring if there exists a Boolean-ring $(A, +, \wedge, 1)$ with identity such that is defined in terms of $+$, $\wedge$ and 1, and for any $b \in B, b \cdot b = b$.

Definition 1.3. A near-ring $(B, +, \cdot)$ is said to be idempotent if $x^2 = x$, for all $x \in B$. i.e. If $(B, +, \cdot)$ is an idempotent ring, then for all $a, b \in B$, $a + a = 0$ and $a \cdot b = b \cdot a$.

Definition 1.4. Compatibility $a \in b$ i.e. "a is compatibility to b" if ab^2=a^2b.

Definition 1.5. Let $A = (\cdots, a, b, c, \cdots)$ be a set of pairwise compatible elements of an associate ring R. Let A be maximal in the sense that each element of A is compatible with

every other element of A and no other such elements may be found in R. Then A is said to be a maximal compatible set or a maximal set.

Definition 1.6. If a sub-direct sum R of domains has an identity, and if R has the property that with each element a, it contains also the associated idempotent a^0 of a, then R is called an associate subdirect sum or an associate ring.

Definition 1.7. If the maximal set A contains an element u which has the property that $a < u$, for all $a \in A$, then u is called the uni-element of A.

Definition 1.8. Left zero divisors are right zero divisors, if $ab = 0$ implies $ba = 0$.

Now we have introduced a new definition by [3].

Definition 1.9. A Boolean-near-ring B is said to be Samarandache-Boolean-near-ring whose proper subset A is a Boolean-ring with respect to same induced operation of B.

Theorem 1.1. A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset A , is a maximal set with uni-element in an associate ring R, with identity under suitable definitions for $(B, +, \dot{})$ with corresponding lattices $(A, \leq)$ $(A, <)$ and

$a \vee b = a + b - 2a^0 b = (a \cup b) - (a \cap b)$,

$a \wedge b = a \cap b = a^0 b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof. Given $(B, \vee, \wedge)$ is a Boolean-near-ring whose proper subset $(A, \vee, \wedge)$ is a maximal set with uni-element in an associate ring R, and if the maximal set A is also a subset of B.

Now to prove that B is Smarandache-Boolean-near-ring. It is enough to prove that the proper subset A of B is a Boolean-ring. Let a and b be two constants of A, if a is compatible to b, we define $a \wedge b$ as follows:

If $a_i = b_i$ in the i-component, let $(a \wedge b)_i = 0_i$;

if $a_i \neq b_i$, then since a∼b precisely one of these is zero;

if $a_i = 0$, let $(a \wedge b)_i = b_i \neq 0$;

if $b_i = 0$, let $(a \wedge b)_i = a_i \neq 0$.

It is seen that if $a \wedge b$ belongs to the associate ring R, then $a \wedge b < u$, where u is the uni-element of A, and therefore, $a \wedge b \in A$.

Consider $a \wedge b = a + b - 2a^0 b$:

If in the i-component, $0 \neq a_i$-b_i, then since $(a^0)_i = 1_i = (b^o)_i$, we have $(a + b - 2a^0 b)_i = 0_i$;

if $0_i = a_i = b_i$, then $(a^0)_i = 0$ and $(b^0)_i = 1$, whence, $(a + b - 2a^0 b)_i = b_i$;

if $a_i \neq 0$ and b_i=0 then $(a + b - 2a^0 b) = 0_i$.

Therefore $a \wedge b \in A$, the maximal set.

Similarly, the element $a \wedge b = a \cap b = a^0 b = ab^0 = glb(a, b)$ has defined and shown to belongs to A as the $glb(a, b)$. Now let us show that $(A, \vee, \wedge)$ is a Boolean-ring. Firstly, $a \vee a = 0$, since $a_i = a_i$ in every i-component, whence $(a \vee a)_i$ vanishes, by our definition of '$\vee$'. Secondly $a \wedge a = a \cap a = a^0 a = a$, and so a is idempotent under $\wedge$. We have shown that A is closed under $\wedge$ is $\vee$, and associativity is a direct verification, and each element is itself inverse under $\wedge$.

To prove associativity under $\wedge$:

For $a \wedge (b \wedge c) = a^0 (b \wedge c) = a^0 (b^0 c) = a^0 (bc^0) = (a^0 b) c^0 = (a \wedge b)^0 c = (a \wedge b) \wedge c$
$\Rightarrow a \wedge (b \wedge c) = (a \wedge b) \wedge c$, for all $a, b, c \in R$.

For distributivity of $\wedge$ over $\wedge$, let c be an arbitrary element in A.

Now $c \wedge (a \vee b) = c^0(a \vee b) = c^0(a \cup b) - c^0(a \cap b) = (c^0 a \cup c^0 b) - c^0 a^0 b = c^0 a + c^0 b - c^0 a^0 b - c^0 a^0 b = c^0 a + c^0 b - 2c^0 a^0 b = (c \wedge a) \vee (c \wedge b) \Rightarrow c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$. Hence $(A, \vee, \wedge)$ is a Boolean-ring.

It follows that the proper subset A, a maximal set of B forms a Boolean ring. B is a Boolean-near-ring, whose proper subset is a Boolean-ring, then by definition, B is a Smarandache-Boolean-near-ring.

Theorem 1.2. A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset $(A, +, \wedge, 1)$ is an associate ring in which the relation of compatibility is transitive for non-zero elements with identity under suitable definitions for $(B, +, \cdot)$ with corresponding lattices $(A, \leq)$ $(A, <)$ and

$a \vee b = a + b - 2a^0 b = (a \cup b) - (a \cap b)$,

$a \wedge b = a \cap b = a^0 b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof. Assume that $(B, +, \cdot)$ be Boolean- near-ring having a proper subset A is an associate ring in which the relation of compatibility is transitive for non-zero elements.

Now to prove that B is a Smarandache-Boolean-near-ring, i.e., to prove that if the proper subset of B is a Boolean-ring, then by definition B is Smarandache-Boolean-near-ring. We have 0 is compatible with all elements, whence all elements are compatible with A and therefore, are idempotent.

Then assume that transitivity holds for compatibility of non-zero elements. It follows that non-zero elements from two maximal sets cannot be compatible (much less equal), and hence, except for the element 0, the maximal sets are disjoint.

Let a be a arbitrary, non-zero element of R. If a is a zero-divisor of R, then the idempotent element $A - a^0 \neq 0$. Further $A - a^0$ belongs to the maximal set generated by the non-zero divisor $a' = a + A - a^0$, since it is $(A - a^0)a' = (A - a^0)(a + A - a^0) = (A - a^0) = (A - a^0)^2$ i.e. $A - a^0 < a'$. Since also $a < a'$ and $a \sim A - a^0$, therefore, a is idempotent. i.e. All the zero-divisors of R are idempotent which is a maximal set then by theorem 1 and by definition A is a Boolean-ring. Then by definition, B is Smarandache-Boolean-near-ring.

Theorem 1.3. A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset A, the set A of idempotent elements of a ring R, with suitable definitions for $\vee$ and $\wedge$,

$a \vee b = a + b - 2a^0 b = (a \cup b) - (a \cap b)$,

$a \wedge b = a \cap b = a^0 b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof.

Assume that the set A of idempotent elements of a ring R, which is also a subset of B. Now to prove that B is a Smarandache-Boolean-near-ring. It is sufficient to prove that the set A of idempotent elements of a ring R with identity forms a maximal set in R with uni-element. By the definition of compatible, then we have every element of R is compatible with every other idempotent element. If $a \in R$ is not idempotent then, $a^2 \cdot 1 \neq a \cdot 1^2$, since the definition of compatible. Hence no non-idempotent can belong to this maximal set. Thus the set A is idempotent element of R with identity forms a maximal set in R whose uni-element is the identity of R, by theorem 1 and by definition. A, a maximal set of B forms a Boolean ring

Then by definition, it concludes that B is Smarandache-Boolean-near-ring.

Theorem 1.4. A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset, having a non-zero divisor of A, as an associate ring, with suitable definitions for $\vee$ and $\wedge$,

$a \vee b = a + b - 2a^0b = (a \cup b) - (a \cap b)$

$a \wedge b = a \cap b = a^0b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof. Let B is Boolean-near-ring whose proper subset having a non-zero divisor of associate ring A.

Now to prove that B Smarandache-Boolean-near-ring. It is enough to prove that every non-divisor of A determines uniquely a maximal set of A with uni-element.

Let a be the uni-element of a maximal set A then we have $b < a$, for $b \in A$.

Consider all the elements of A in whose sub-direct display one or more component a_i duplicate the corresponding component u_i of u, the other components of a being zeros, i.e., all the element a such that $a < u$, becomes u is uni-element. Clearly, these elements are compatible with each other and together with u form a maximal set in A, for which u is the uni-element. Hence A is a maximal set with uni-element and by theorem 1 and definition A, a maximal set of B forms a Boolean ring.

Then by definition, B is Smarandache-Boolean-near-ring.

Theorem 1.5. A Boolean-near-ring $(B, \vee, \wedge)$ is having the proper subset A, associate ring is of the form $A = u_J$, where u is a non-zero of A and J is the set of idempotent elements of A, with suitable definitions for $\vee$ and $\wedge$,

$a \vee b = a + b - 2a^0b = (a \cup b) - (a \cap b)$,

$a \wedge b = a \cap b = a^0b = ab^0$.

Then B is a Smarandache-Boolean-near-ring.

Proof. Assume that the proper subset A of a Boolean-near-ring B is of the form $A = u_J$, where u is non-zero divisor of A and J is the set of idempotent elements of A. Now to prove B is Smarandache-Boolean-near-ring. It is enough to prove that A is a maximal set with uni-element.

(i) It is sufficient to show that the set u_J is a maximal set having u as its uni-element.

(ii) If b belongs to the maximal set determined by u, then b has the required form $b = u_e$, for some $e \in J$.

Proof of (i). It is seen that $u_e \sim u_f$ i.e. u_e is compatible to u_f with uni-element u, for all $e, f \in J$, since idempotent belongs to the center of A. Also, $u_e < u$, since $u_e \cdot u = u_e^2 = (u_e)^2$.

Proof of (ii). We know that A is an associate ring, the associated idempotent a^0 of a has the property: if $a \sim$b then $a^0b = ab^0 = b^0a = ba^0$; if $a \in A_u$, then since $a < u$ and $u^0 = 1$, we have $A = u^0a = au^0 = a^0u$, for all $a^0 \in J$.

Hence A is a maximal set with uni-element of of B by suitable definition and by theorem 1 then we have A is a Boolean-ring. Then by definition, B is Smarandache-Boolean-near-ring.

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Some indefinite integrals involving certain polynomial

Salahuddin

Mewar University, Gangrar, Chittorgarh (Rajasthan), India
E-mail: vsludn@gmail.com, sludn@yahoo.com

Abstract In this paper we have established two indefinite integrals involving Lucas polynomials, Fibonacci polynomials, Gegenbauer polynomials and Hermite polynomials. The results represent here are assumed to be new.

Keywords Gegenbauer polynomials, hermite polynomials; Fibbonacci polynomials, Lucas pol -ynomials, polylogarithm function, hypergeometric function.
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§1. Introduction and preliminaries

Definition 1.1. Gegenbauer polynomials or ultraspherical polynomials $C_n^{\alpha}(x)$ are orthogonal polynomials on the interval [-1,1] with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$. They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials. They are named for Leopold Gegenbauer.

Gegenbauer polynomials are particular solutions of the Gegenbauer differential equation

$$(1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0.$$

When $\alpha = \frac{1}{2}$, the equation reduces to the Legendre equation, and the Gegenbauer polynomials reduce to the Legendre polynomials.

They are given as Gaussian hypergeometric series in certain cases where the series is in fact finite

$$C_n^{\alpha}(z) = \frac{(2\alpha)_n}{n!}\,{}_2F_1\left(-n, (2\alpha+n); \alpha+\frac{1}{2}; \frac{1-z}{2}\right).$$

They are special cases of the Jacobi polynomials

$$C_n^{\alpha}(z) = \frac{(2\alpha)_n}{(\alpha+\frac{1}{2})_n} P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x).$$

One therefore also has the Rodrigues formula

$$C_n^{\alpha}(z) = \frac{(-2)^n}{n!}\frac{\Gamma(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(\alpha)\Gamma(2n+2\alpha)}(1-x^2)^{-\alpha+\frac{1}{2}}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-\frac{1}{2}}\right].$$

Definition 1.2. The Pochhammer's symbol is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)\cdots(b+k-1), & \text{if } k = 1, 2, 3, \cdots; \\ 1 & \text{, if } k = 0; \\ k! & \text{, if } b = 1,\ k = 1, 2, 3, \cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for gamma function.

Definition 1.3. In mathematics, the Bernoulli polynomials occur in the study of many special functions and in particular the Riemann zeta function and the Hurwitz zeta function. This is in large part because they are an Appell sequence, i.e., a Sheffer sequence for the ordinary derivative operator. Unlike orthogonal polynomials, the Bernoulli polynomials are remarkable in that the number of crossings of the x-axis in the unit interval does not go up as the degree of the polynomials goes up. In the limit of large degree, the Bernoulli polynomials, appropriately scaled, approach the sine and cosine functions.

The Bernoulli polynomials $B_n(x)$ admit a variety of different representations. Which among them should be taken to be the definition may depend on one's purposes.

Definition 1.4. Explicit formula is defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{n-k} B_k x^{n-k},$$

for $n \geq 0$, where B_k are the Bernoulli numbers.

Definition 1.5. In mathematics, the Hermite polynomials are a classical orthogonal polynomial sequence that arise in probability, such as the Edgeworth series; in combinatorics, as an example of an Appell sequence, obeying the umbral calculus; in numerical analysis as Gaussian quadrature; and in physics, where they give rise to the eigenstates of the quantum harmonic oscillator. They are also used in systems theory in connection with nonlinear operations on Gaussian noise. They are named after Charles Hermite (1864) although they were studied earlier by Laplace (1810) and Chebyshev (1859).

There are two different standard ways of normalizing Hermite polynomials:

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}},$$

(the "probabilists' Hermite polynomials"), and

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = e^{\frac{x^2}{2}} \left(x - \frac{d}{dx}\right)^n e^{-\frac{x^2}{2}},$$

(the "physicists' Hermite polynomials").

These two definitions are not exactly equivalent; either is a rescaling of the other, to wit

$$H_n(x) = 2^{\frac{n}{2}} He_n(\sqrt{2}x), \quad He_n(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right).$$

The notation He and H is that used in the standard references Tom H. Koornwinder, Roderick S. C. Wong, and Roelof Koekoek et al(2010), and Abramowitz & Stegun. The polynomials He_n are sometimes denoted by H_n, especially in probability theory, because $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the

probability density function for the normal distribution with expected value 0 and standard deviation 1.

The first eleven probabilists' Hermite polynomials are:

$$He_0(x) = 1,$$

$$\begin{aligned} He_1(x) &= x, \\ He_2(x) &= x^2 - 1, \\ He_3(x) &= x^3 - 3x, \\ He_4(x) &= x^4 - 6x^2 + 3, \\ He_5(x) &= x^5 - 10x^3 + 15x, \\ He_6(x) &= x^6 - 15x^4 + 45x^2 - 15, \\ He_7(x) &= x^7 - 21x^5 + 105x^3 - 105x, \\ He_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105, \\ He_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x, \\ He_{10}(x) &= x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945, \end{aligned}$$

and the first eleven physicists' Hermite polynomials are:

$$H_0(x) = 1,$$

$$\begin{aligned} H_1(x) &= 2x, \\ H_2(x) &= 4x^2 - 2, \\ H_3(x) &= 8x^3 - 12x, \\ H_4(x) &= 16x^4 - 48x^2 + 12, \\ H_5(x) &= 32x^5 - 160x^3 + 120x, \\ H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120, \\ H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x, \\ H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680, \\ H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x, \\ H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240. \end{aligned}$$

Definition 1.6. The sequence of Lucas polynomials is a sequence of polynomials defined by the recurrence relation

$$L_n(x) = \begin{cases} 2x^0 = 2 & , \quad \text{if } \ n = 0; \\ 1x^1 = x & , \quad \text{if } \ n = 1; \\ x^1 L_{n-1}(x) + x^0 L_{n-2}(x) & , \quad \text{if } \ n \geq 2. \end{cases}$$

The first few Lucas polynomials are:

$$L_0(x) = 2,$$

$$L_1(x) = x,$$

$$L_2(x) = x^2 + 2,$$

$$L_3(x) = x^3 + 3x,$$

$$L_4(x) = x^4 + 4x^2 + 2.$$

Definition 1.7. In mathematics, the Fibonacci polynomials are a polynomial sequence which can be considered as a generalization of the Fibonacci numbers.

These Fibonacci polynomials are defined by a recurrence relation:

$$F_n(x) = \begin{cases} 0 & , \quad \text{if } \ n = 0; \\ 1 & , \quad \text{if } \ n = 1; \\ xF_{n-1}(x) + F_{n-2}(x) & , \quad \text{if } \ n \geq 2. \end{cases}$$

The first few Fibonacci polynomials are:

$$F_0(x) = 0,$$

$$F_1(x) = 1,$$

$$F_2(x) = x,$$

$$F_3(x) = x^2 + 1,$$

$$F_4(x) = x^3 + 2x.$$

Definition 1.8. The polylogarithm (also known as Jonquie's function) is a special function $Li_s(z)$ that is defined by the infinite sum, or power series

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

It is in general not an elementary function, unlike the related logarithm function. The above definition is valid for all complex values of the order s and the argument z where $\mid z \mid < 1$.

Definition 1.9. Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B \left[\begin{array}{lll} a_1, a_2, \cdots, a_A & ; & \\ & & z \\ b_1, b_2, \cdots, b_B & ; & \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!},$$

or

$${}_AF_B \left[\begin{array}{lll} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array} \right] \equiv {}_AF_B \left[\begin{array}{lll} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array} \right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!},$$

where denominator parameters $b_1, \ b_2, \ \cdots, \ b_B$ are neither zero nor negative integers and $A, \ B$ are non-negative integers.

§2. Main indefinite integrals

$$
\begin{aligned}
&\int \frac{\sin x \ H_2(x)L_1(x)F_1(x)}{\sqrt{1-\sin x}}\ \mathrm{dx} \\
= \ & \frac{1}{\sqrt{1-\sin x}}(2+2\iota)\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)\times\Big[(-1)^{\frac{3}{4}}\Big\{96x^2 Li_3\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-96x^2 Li_3\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big) \\
&-4\iota(4x^2-1)xLi_2\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+4\iota(4x^2-1)xLi_2\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+384\iota xLi_4\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big) \\
&-384\iota xLi_4\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-8Li_3\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+8Li_3\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-768Li_5\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big) \\
&+768Li_5\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-2x^4\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+2x^4\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+x^2\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big) \\
&-x^2\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big\}-(1-\iota)(2x^4-16x^3-97x^2+388x+776)\sin\frac{x}{2} \\
&+(-1+\iota)(2x^4+16x^3-97x^2-388x+776)\cos\frac{x}{2}\Big]+\text{Constant.}
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{\cos x \ L_4(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} = \frac{1}{5\sqrt{1-\cos x}}\sin\frac{x}{2}\Big[80\ \iota x^3 Li_2\Big(e^{-\frac{\iota x}{2}}\Big)+480x^2Li_3\Big(e^{-\frac{\iota x}{2}}\Big)-480x^2Li_3\Big(-e^{\frac{\iota x}{2}}\Big) \\
&+80\iota(x^2+2)xLi_2\Big(-e^{\frac{\iota x}{2}}\Big)-160\iota xLi_2\Big(e^{\frac{\iota x}{2}}\Big)-1920\iota xLi_4\Big(e^{\frac{-\iota x}{2}}\Big)-1920\iota xLi_4\Big(-e^{\frac{\iota x}{2}}\Big) \\
&-320Li_3\Big(-e^{\frac{\iota x}{2}}\Big)+320Li_3\Big(e^{\frac{\iota x}{2}}\Big)-3840Li_5\Big(e^{-\frac{\iota x}{2}}\Big)+3840Li_5\Big(-e^{\frac{\iota x}{2}}\Big)+\iota x^5+10x^4\log(1-e^{-\frac{\iota x}{2}}) \\
&-10x^4\log(1+e^{\frac{\iota x}{2}})+20x^4\cos\frac{x}{2}-160x^3\sin\frac{x}{2}+40x^2\log(1-e^{\frac{\iota x}{2}})-40x^2\log(1+e^{\frac{\iota x}{2}}) \\
&-880x^2\cos\frac{x}{2}+3520x\sin\frac{x}{2}+7080\cos\frac{x}{2}+20\log\Big(\tan\frac{x}{4}\Big)-16\iota\pi^5\Big]+\text{Constant.}
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{\cosh^3 x \ L_3(x)\ C_3(x)}{\sqrt{1-\cosh^2 x}}\ \mathrm{dx} \\
= \ & \frac{1}{1680\sqrt{-\sinh^2 x}}\Big[\sinh x\Big\{67200x^3Li_4(e^{2x})-100800x^2Li_5(e^{2x})+3360(4x^4+6x^2-3)xLi_2(e^{2x}) \\
&-1680(20x^4+18x^2-3)Li_3(e^{2x})+30240xLi_4(e^{2x})+100800xLi_6(e^{2x})-15120Li_5(e^{2x}) \\
&-50400Li_7(e^{2x})-640x^7+4480x^6\log(1-e^{2x})+1120x^6\cosh 2x-2016x^5-3360x^5\sinh 2x \\
&+10080x^4\log(1-e^{2x})+10920x^4\cosh 2x+3360x^3-21840x^3\sinh 2x-10080x^2\log(1-e^{2x}) \\
&+30240x^2\cosh 2x-30240x\sinh 2x+15120\cosh 2x+5\iota\pi^7-63\iota\pi^5-420\iota\pi^3\Big\}\Big]+\text{Constant.}
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{\cos x \ L_1(x)\ H_1(x)\ F_1(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} \\
= \ & \frac{1}{2\sqrt{1-\cos x}}\sin\frac{x}{2}\Big[48\ \iota x^2 Li_2(e^{-}\frac{\iota x}{2})+48\iota x^2Li_2(-e^{\frac{\iota x}{2}})+192xLi_3(e^{-}\frac{\iota x}{2})-192xLi_3(-e^{\frac{\iota x}{2}}) \\
&-384\iota Li_4(e^{-}\frac{\iota x}{2})-384\iota Li_4(-e^{\frac{\iota x}{2}})+\iota x^4+8x^3\log(1-e^{-}\frac{\iota x}{2})-8x^3\log(1+e^{\frac{\iota x}{2}}) \\
&+16x^3\cos\frac{x}{2}-96x^2\sin\frac{x}{2}+768\sin\frac{x}{2}-384x\cos\frac{x}{2}-8\iota\pi^4\Big]+\text{Constant.}
\end{aligned}
$$

$$\begin{aligned}
&\int \frac{\cosh x \;\; H_1(x) \;\; B_1(x)}{\sqrt{1-\cosh x}} \text{ dx} \\
= \;& \frac{2}{\sqrt{1-\cosh x}} \sinh\frac{x}{2}\Big[(8x-2)Li_2(-e^{\frac{-x}{2}})+(2-8x)Li_2(e^{\frac{-x}{2}}) \\
&+16Li_3(-e^{\frac{-x}{2}})-16Li_3(e^{\frac{-x}{2}})+2x^2\log(1-e^{\frac{-x}{2}})-2x^2\log(1+e^{\frac{-x}{2}})+4x^2\cosh\frac{x}{2} \\
&-x\log(1-e^{\frac{-x}{2}})+x\log(1+e^{\frac{-x}{2}})-16x\sinh\frac{x}{2}+4\sinh\frac{x}{2}-2x\cosh\frac{x}{2}+32\cosh\frac{x}{2}\Big] \\
&+\text{Constant}.
\end{aligned}$$

$$\begin{aligned}
&\int \frac{\cosh x \;\; F_2(x)}{\sqrt{1-\cos x}} \text{ dx} \\
= \;& -\frac{1}{\sqrt{1-\cos x}}\Big(\frac{8}{25}-\frac{6\iota}{25}\Big)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\Big[2e^{2x}{}_3F_2\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota, \\
&\frac{1}{2}-\iota;e^{\iota x}\Big)+2e^{\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\Big)-(2-\iota)xe^{2x}{}_2F_1\Big(-\frac{1}{2}-\iota,,1;\frac{1}{2}-\iota;e^{\iota x}\Big) \\
&+(2-\iota)xe^{\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,,1;\frac{3}{2}+\iota;e^{\iota x}\Big)+(2-\iota)xe^{2x}-2e^{2x}\Big]+\text{Constant}.
\end{aligned}$$

$$\begin{aligned}
&\int \frac{\cos x \;\; L_3(x)}{\sqrt{1-\cos x}} \text{ dx} = \frac{1}{4\sqrt{1-\cos x}}\sin\frac{x}{2}\Big[48\;\iota x^2 Li_2\Big(e^{-\frac{\iota x}{2}}\Big)+48\iota(x^2+1)Li_2\Big(-e^{\frac{\iota x}{2}}\Big) \\
&+192xLi_3\Big(e^{-\frac{\iota x}{2}}\Big)-192xLi_3\Big(-e^{\frac{\iota x}{2}}\Big)-48\iota Li_2\Big(e^{\frac{\iota x}{2}}\Big)-384\iota Li_4\Big(e^{-\frac{\iota x}{2}}\Big)-384\iota Li_4\Big(-e^{\frac{\iota x}{2}}\Big) \\
&+\iota x^4+8x^3\log\Big(1-e^{\frac{-\iota x}{2}}\Big)-8x^3\log\Big(1+e^{\frac{\iota x}{2}}\Big)+16x^3\cos\frac{x}{2}-96x^2\sin\frac{x}{2}+24x\log\Big(1-e^{\frac{\iota x}{2}}\Big) \\
&-24x\log\Big(1+e^{\frac{\iota x}{2}}\Big)+672\sin\frac{x}{2}-336x\cos\frac{x}{2}-8\iota\pi^4\Big]+\text{Constant}.
\end{aligned}$$

$$\begin{aligned}
&\int \frac{\sin x \;\; F_{13}(x)}{\sqrt{1-\cosh x}} \text{ dx} \\
= \;& \frac{233}{25\sqrt{1-\cosh x}}e^{-\iota x}(e^x-1)\Big[-(8+6\iota)\;{}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big) \\
&-(8-6\iota)e^{2\iota x}\;{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;\cosh x+\sinh x\Big) \\
&+5x\Big\{(2-\iota)\;{}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big) \\
&+(2+\iota)e^{2\iota x}\;{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;\cosh x+\sinh x\Big)\Big\}\Big]+\text{Constant}.
\end{aligned}$$

$$\begin{aligned}
&\int \frac{\sin x \;\; F_{19}(x)}{\sqrt{1-\cosh x}} \text{ dx} \\
= \;& \frac{4181}{25\sqrt{1-\cosh x}}e^{-\iota x}(e^x-1)\Big[-(8+6\iota)\;{}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big) \\
&-(8-6\iota)e^{2\iota x}\;{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;\cosh x+\sinh x\Big)+ \\
&5x\Big\{(2-\iota)\;{}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big) \\
&+(2+\iota)e^{2\iota x}\;{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;\cosh x+\sinh x\Big)\Big\}\Big]+\text{Constant}.
\end{aligned}$$

$$
\begin{aligned}
& \int \frac{\cosh x\ L_2(x)\ C_2(x)}{\sqrt{1-\cosh x}}\ \mathrm{dx} \\
= \quad & \frac{2}{5\sqrt{1-\cosh x}} \sinh \frac{x}{2} \Big[80x^3 Li_2(e^{\frac{x}{2}}) + 480x^2 Li_3(-e^{\frac{-x}{2}}) - 480x^2 Li_3(e^{\frac{x}{2}}) \\
& +20(4x^2+3)x Li_2(-e^{\frac{-x}{2}}) - 60x Li_2(e^{\frac{-x}{2}}) \\
& +1920x Li_4(-e^{\frac{-x}{2}}) + 1920x Li_4(e^{\frac{x}{2}}) + 120 Li_3(-e^{\frac{-x}{2}}) \\
& -120 Li_3(e^{\frac{-x}{2}}) + 3840 Li_5(-e^{\frac{-x}{2}}) - 3840 Li_5(e^{\frac{x}{2}}) - x^5 \\
& -10x^4 \log(1+e^{\frac{-x}{2}}) + 10x^4 \log(1-e^{\frac{x}{2}}) + 20x^4 \cosh \frac{x}{2} - 160x^3 \sinh \frac{x}{2} \\
& +15x^2 \log(1-e^{\frac{-x}{2}}) - 15x^2 \log(1+e^{\frac{-x}{2}}) + 990x^2 \cosh \frac{x}{2} \\
& -3960x \sinh \frac{x}{2} + 7900 \cosh \frac{x}{2} - 10 \log\left(\tanh \frac{x}{4}\right) - 16\iota\pi^5 \Big] + \text{Constant}.
\end{aligned}
$$

§3. Derivation of the integrals

Involving the method of same type of [6] , one can derive the integrals.

§4. Conclusion

In our work we have established certain indefinite integrals involving Fibonacci polynomials, Lucas polynomials, Bernoulli polynomials and Hermite polynomials. We hope that the development presented in this work will stimulate further interest and research in this important area of Mathematics.

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Three Classes of Exact Solutions to Klein–Gordon–Schrödinger Equation [1]

Hongming Xia †, Wansheng He ‡, Linxia Hu ♯ and Zhongshe Gao ♮

College of Mathematics and Statistics, Tianshui Normal University,
Tianshui-741 001, P. R. China
E-mail: xia_hm@sina.com

Abstract Basing on the priori assumption principle, the main goal of this paper is to propose the rational expansion method that can be used to handle the exact solution of the nonlinear partial differential equation. It is extension of tanh function method. As an application, three classes of exact solutions to Klein-Gordon-Schrödinger equation are obtained.

Keywords Klein-Gordon-Schrödinger equation, exact solution, tanh function method, rational expansion method.

§1. Introduction

The investigation of exploring the exact traveling wave solutions of nonlinear mathematical physics equations plays an important role in the study of the solitary wave solution. Up to now, a variety of methods, such as inverse scattering method, Bäcklund, transformation method, Hirota, bilinear method, homogeneous balance method and tanh method [1-5]. The tanh method is one of the most direct and effective algebraic for computing the exact traveling wave solution. In this paper, we extend the method by replacing tanh function with some other functions $f(x)$, such as polynomial function, trigonometric and elliptical function, where the choice of the function $f(x)$ is different according to equation. Similarly to the steps introduced in [6], we can get the exact solution for equation, namely, assume that after simplifying the solution of the simplified nonlinear PDE, equation has the following form.

$$u(\xi) = \frac{a_m f(\xi)^m + a_{m-1} f(\xi)^{m-1} + \cdots + a_0}{b_m f(\xi)^m + b_{m-1} f(\xi)^{m-1} + \cdots + b_0}. \tag{1}$$

Then the solution can be obtained by using the above method and this develops the tanh method.

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§2. The extended tanh method

Consider the nonlinear PDE

$$F(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{2}$$

with two variables $x,\ t$. Let $u(x,t) = U(\xi)$, $\xi = x - Vt$ be its travelling wave solutions, where the wave velocity V is a coefficient to be determined later, then equation (2) can be simplified to a nonlinear ODE

$$G(U, U^{'}, U^{''}, \cdots) = 0. \tag{3}$$

Assume that the solution of equation (3) has the form of (1). Substituting (1) into (3), then m can be determined by balancing the linear terms of the highest order in the resulting equation with the highest order nonlinear terms. This will give a system of algebra equations involving $a_i, b_i (i = 0, 1, \cdots, 2m)$ and α.

Let the function $f(x)$ be $e^{\alpha\xi}$, then the main steps for achieving the above method is as following.

Step 1 Suppose that equation (2) has the wave traveling solution $u(x,t) = U(\xi) = U(x - Vt)$. Then equation (2) can be simplified to a nonlinear ODE (3);

Step 2 assume that the solution of (3) has the form of (1), then m can be determined by using the balance method;

Step 3 substituting (1) into (3), we get a rational fractional equation

$$P(e^{\alpha\xi}) = 0, \tag{4}$$

which is just a polynomials of exponential $e^{\alpha\xi}$;

Step 4 collecting all the terms with the same power of $e^{\alpha\xi}$ yields a set of algebraic system for $a_i, b_i (i = 0, 1, \cdots, 2m)$, α, where $a_i, b_i (i = 0, 1, \cdots, 2m), \alpha$ are coefficients to be determined later;

Step 5 applying some mathematical package, for example, Mathematica, Maple, and etc., we can deal with the above tedious algebra equations and output directly the required solution. Further, substituting the solution into (1) and letting, $\xi = x - Vt$, we can get the exact solution of equation (2).

§3. The exact solutions of Klein-Gordon-Schrödinger equation

KGS equations

$$\begin{cases} i\psi_t + \dfrac{1}{2}\Delta\phi = -\phi\psi, \\ \phi_{tt} - \Delta\phi + m^2\phi = |\psi|^2. \end{cases} \tag{5}$$

is a classical model which reflects the interaction of nucleon-field and meson-field, where Δ is a n, dimensional Laplace operator, ψ is scalar complete nucleon-field and ϕ is meson-field [7]. In

[8] the author had proved that the steady-state solution of equations (5) has the form

$$(\psi(x,t),\phi(x,t)) = (e^{i\omega t}u(x), v(x)), \tag{6}$$

where $x \in R^3$, $\omega \in R$. We will take into account the exact solution of equation (5) by using the technique basing on the expansion of the rational function method.

Here we only focus on the (1+1) dimensional equation, the discussion for (1+n) dimensional equation is similar and is omitted. Consider the following equation

$$\begin{cases} i\psi_t + \dfrac{1}{2}\phi_{xx} = -\phi\psi, \\ \phi_{tt} - \phi_{xx} + m^2\phi = |\psi|^2. \end{cases} \tag{7}$$

where ψ is complex. Assume that

$$\psi = e^{i\eta}u(x,t), \tag{8}$$

where $\eta = \alpha x + \beta t$ and α, β are coefficients which will be determined later. Substituting (8) into equations (7) yields

$$\begin{cases} u_t + \alpha u_x = 0, \\ u_{tt} + 2u\phi - (\alpha^2 + 2\beta)u = 0, \\ \phi_{tt} - \phi_{xx} + m^2\phi - u^2 = 0. \end{cases} \tag{9}$$

Let $u(x,t) = U(\xi) = U(kx + \omega t)$, $V(\xi) = \phi(kx + \omega t)$, where k and ω are coefficients to be determined later. Then we obtain a simplified ODE.

$$\begin{cases} (\dfrac{1}{\omega} + \dfrac{\alpha}{k})U_\xi = 0, \\ \dfrac{1}{\omega^2}U_{\xi\xi} + 2UV - (\alpha^2 + 2\beta)U = 0, \\ \dfrac{k^2 - \omega^2}{\omega^2 k^2}V_{\xi\xi} + m^2 V - U^2 = 0. \end{cases} \tag{10}$$

Suppose that the solution, $U(\xi)$ and $V(\xi)$, for equations (10) are expansion of a second order differential function, then we can get the exact solution of equations (9) basing on the homogeneous balance principle.

§4. Main result

1. Suppose that the solution for equations (10) has the following form

$$\begin{cases} U(\xi) = \dfrac{a_m\xi^m + a_{m-1}\xi^{m-1} + \cdots + a_0}{b_m\xi^m + b_{m-1}\xi^{m-1} + \cdots + b_0}, \\ V(\xi) = \dfrac{c_m\xi^m + c_{m-1}\xi^{m-1} + \cdots + c_0}{d_m\xi^m + d_{m-1}\xi^{m-1} + \cdots + d_0}, \end{cases} \tag{11}$$

where $a_i,\ b_i,\ c_i,\ d_i$ $(i = 0,\ 1,\ \cdots,\ m)$ are coefficients to be determined later. By comparing the highest derivative term with the highest nonlinear term in homogeneous balance equations,

we get $m = 2$. Thus equation (10) has the solution of the following form

$$\begin{cases} U(\xi) = \dfrac{a\xi^2 + b\xi + c}{f\xi^2 + g\xi + h}, \\ V(\xi) = \dfrac{o\xi^2 + p\xi + q}{r\xi^2 + s\xi + w}, \end{cases} \tag{12}$$

where $a,\ b,\ c,\ f,\ g,\ h,\ o,\ p,\ q,\ r,\ s,\ w$ are coefficients to be determined later. Substituting (12) into (10), we get a rational fractional equations which is just for ξ. Collecting all the terms with the same power ξ^k and letting their coefficients be zero yields a set of algebraic system for, $a,\ b,\ c,\ f,\ g,\ h,\ o,\ p,\ q,\ r,\ s,\ w,\ k,\ \alpha,\ \beta,\ \omega$. Using Maple package, we can deal with the above algebra equations and output directly the required solution. Further, substituting the solution into (12) and (8), we obtain the following exact solution for equation (7).

Case 1. $o = a = b = p = f = 0,\ s = \dfrac{rh}{2g},\ r = r,\ m = m,\ k = k,\ g = g,\ h = h,\ w = \dfrac{h^2 r}{g^2},\ \beta = \pm 1,\ c = igkm,\ \omega = k,\ \alpha = -1,\ q = -k^2 r,\ \psi_1(x,t) = \dfrac{ie^{(-x\pm t)i} gkm}{g(kx + kt) + h},\ \phi_1(x,t) = \dfrac{-k^2 g^2}{(g(kx + kt) + h)^2}$.

Case 2. $\beta = i,\ q = -k^2 r,\ b = ifkm,\ f = f,\ a = h = c = o = p = 0,\ w = \dfrac{rg^2}{f^2},\ s = \dfrac{2gr}{f},\ \omega = -k,\ \alpha = 1,\ m = m,\ r = r,\ k = k,\ g = g,\ \psi_2(x,t) = \dfrac{ie^{ix-t} fkm}{f(kx - kt) + g},\ \phi_2(x,t) = \dfrac{-k^2 f^2}{(f(kx - kt) + g)^2}$.

Case 3. $\beta = i,\ q = -k^2 r,\ b = ifkm,\ f = f,\ a = h = w = s = o = p = 0,\ c = ikmg,\ \omega = -k,\ \alpha = 1,\ m = m,\ r = r,\ k = k,\ g = g,\ \psi_3(x,t) = \dfrac{ie^{ix-t} m}{x - t},\ \phi_3(x,t) = \dfrac{-1}{(x - t)^2}$.

Case 4. $q = -k^2 r,\ c = \dfrac{ikmg}{2},\ b = ifkm,\ f = f,\ a = o = p = 0,\ \beta = \pm 1,\ w = \dfrac{rg^2}{4f^2},\ s = \dfrac{gr}{f},\ h = \dfrac{g^2}{4f},\ \alpha = -1,\ \omega = k,\ m = m,\ r = r,\ k = k,\ g = g,\ \psi_4(x,t) = \dfrac{ie^{ix\mp t} m}{x - t}, \phi_4(x,t) = \dfrac{-4k^2 f^2}{(2f(kx + kt) + g)^2}$.

Case 5. $f = f,\ c = c,\ b = b,\ h = \dfrac{-(cf - bg)c}{b^2},\ a = o = p = 0,\ k = \dfrac{ib}{mf},\ \beta = \pm 1,\ w = \dfrac{r(c^2 f^2 - 2bgcf + b^2 g^2)}{b^2 f^2},\ q = \dfrac{b^2 r}{m^2 f^2},\ \omega = \dfrac{ib}{mf},\ s = \dfrac{-2r(cf - bg)}{fb},\ \alpha = -1,\ m = m,\ r = r,\ g = g,\ \psi_5(x,t) = \dfrac{e^{-i(x\mp t)} b^2 m}{ib^2(x + t) - cfm + bgm},\ \phi_5(x,t) = \dfrac{b^4}{(ib^2(x + t) - cfm + bgm)^2}$.

Case 6. $\beta = i,\ f = f,\ h = \dfrac{-(cf - bg)c}{b^2},\ a = o = p = 0, k = \dfrac{ib}{mf},\ \omega = \dfrac{-ib}{mf},\ w = \dfrac{r(c^2 f^2 - 2bgcf + b^2 g^2)}{b^2 f^2},\ q = \dfrac{b^2 r}{m^2 f^2},\ c = c,\ b = b,\ s = \dfrac{-2r(cf - bg)}{fb}, \alpha = 1,\ m = m,\ r = r,\ g = g,\ \psi_6(x,t) = \dfrac{e^{(xi-t)} b^2 m}{ib^2(x - t) - cfm + bgm},\ \phi_6(x,t) = \dfrac{b^4}{(ib^2(x - t) - cfm + bgm)^2}$.

2. Suppose that the solution of equations (10) has the following form

$$\begin{cases} U(\xi)=\dfrac{a_m(\tanh\xi)^m+a_{m-1}(\tanh\xi)^{m-1}+\cdots+a_0}{b_m(\tanh\xi)^m+b_{m-1}(\tanh\xi)^{m-1}+\cdots+b_0}, \\ V(\xi)=\dfrac{c_m(\tanh\xi)^m+c_{m-1}(\tanh\xi)^{m-1}+\cdots+c_0}{d_m(\tanh\xi)^m+d_{m-1}(\tanh\xi)^{m-1}+\cdots+d_0}, \end{cases} \tag{13}$$

where $a_i,\ b_i,\ c_i,\ d_i\ (i=0,\ 1,\ \cdots,\ m)$ are coefficients to be determined later. By comparing the highest derivative term with the highest nonlinear term in homogeneous balance equations of equations (10), we get $m=2$. Thus equation (10) has the solution of the following form

$$\begin{cases} U(\xi)=\dfrac{a(\tanh\xi)^2+b\tanh\xi+c}{f(\tanh\xi)^2+g\tanh\xi+h}, \\ V(\xi)=\dfrac{o(\tanh\xi)^2+p\tanh\xi+q}{r(\tanh\xi)^2+s\tanh\xi+w}, \end{cases} \tag{14}$$

where $a,\ b,\ c,\ f,\ g,\ h,\ o,\ p,\ q,\ r,\ s,\ w$ are coefficients to be determined later. Substituting (14) into equations (10), we get a rational fractional equations which is just for $\tanh\xi$.

Collecting all the terms with the same power of $\tanh\xi$ and letting their coefficients be zero yields a set of algebraic system for $a,\ b,\ c,\ f,\ g,\ h,\ o,\ p,\ q,\ r,\ s,\ w,\ k,\ \alpha,\ \beta,\ \omega$. Using Maple, package, we can deal with the above algebra equations and output directly the required solution. Further, substituting the solution and $\eta=\alpha x+\beta t$, $\xi=kx+\omega t$ into (14) and (8), we obtain the following exact solution for equation (7).

Case 1. $q=\dfrac{-w\left(24\,k^2f^2-g^2\alpha-g^2\beta^2\right)}{2g^2}$, $\omega=-\alpha\ k$, $s=\dfrac{4fw}{g}$, $a=b=m=0$, $f=f$, $c=6$, $k^2\alpha^2f-6$, k^2f, $p=\dfrac{2fw\left(\alpha+2\beta^2\right)}{g}$, $h=\dfrac{g^2}{4f}$, $o=\dfrac{2f^2w\left(\alpha+\beta^2\right)}{g^2}$, $\alpha=\alpha$, $\beta=\beta$, $w=w$, $r=\dfrac{4f^2w}{g^2}$, $k=k$, $g=g$, $\psi_7(x,t)=\dfrac{24e^{i(\alpha x+\beta t)}k^2f^2\left(\alpha^2-1\right)}{\left(2\,f\tanh\left(-kx+\alpha\,kt\right)-g\right)^2}$, $\phi_7(x,t)=\dfrac{\left(-2\,f\tanh\left(-kx+\alpha\,kt\right)+g\right)^2\left(\alpha+\beta^2\right)-24\,k^2f^2}{\left(2\,f\tanh\left(-kx+\alpha\,kt\right)-g\right)^2}$.

Case 2. $h=h$, $w=\dfrac{h^2r}{g^2}$, $\beta=i$, $q=-k^2r$, $f=0$, $a=b=o=p=0$, $\omega=-k$, $s=\dfrac{2hr}{g}$, $c=igkm$, $\alpha=1$, $m=m$, $r=r$, $k=k$, $g=g$, $\psi_8\left(x,t\right)=\dfrac{ie^{ix-t}gkm}{g\tanh(kx-kt)+h}$, $\phi_8\left(x,t\right)=\dfrac{-k^2g^2}{\left(g\tanh(kx-kt)+h\right)^2}$.

Case 3. $\beta=i$, $q=-k^2r$, $c=\dfrac{ikmg}{2}$, $b=ifkm$, $f=f$, $a=o=p=0$, $\omega=-k$, $w=\dfrac{rg^2}{4f^2}$, $s=\dfrac{gr}{f}$, $h=\dfrac{g^2}{4f}$, $\alpha=1$, $m=m$, $r=r$, $k=k$, $g=g$, $\psi_9\left(x,t\right)=\dfrac{2\,ie^{ix-t}fkm}{2\,f\tanh(kx-kt)+g}$, $\phi_9\left(x,t\right)=\dfrac{-4k^2f^2}{\left(2\,f\tanh(kx-kt)+g\right)^2}$.

3. Suppose that the solution of equations (10) has the following form

$$\begin{cases} U(\xi)=\dfrac{a_me^{m\xi}+a_{m-1}e^{(m-1)\xi}+\cdots+a_0}{b_me^{m\xi}+b_{m-1}e^{(m-1)\xi}+\cdots+b_0}, \\ V(\xi)=\dfrac{c_me^{m\xi}+c_{m-1}e^{(m-1)\xi}+\cdots+c_0}{d_m(e^{m\xi}+d_{m-1}e^{(m-1)\xi}+\cdots+d_0}, \end{cases} \tag{15}$$

where $a_i, b_i, c_i, d_i (i = 0, 1, \cdots, m)$ are coefficients to be determined later. By comparing the highest derivative term with the highest nonlinear term in homogeneous balance equations for (10), we get $m = 2$, namely, equation (10) has the solution of the following form

$$\begin{cases} U(\xi) = \dfrac{ae^{2\xi} + be^{\xi} + c}{fe^{2\xi} + ge^{\xi} + h}, \\ V(\xi) = \dfrac{oe^{2\xi} + pe^{\xi} + q}{re^{2\xi} + se^{\xi} + w}, \end{cases} \tag{16}$$

where $a,\ b,\ c,\ f,\ g,\ h,\ o,\ p,\ q,\ r,\ s,\ w$ are coefficients to be determined later. Substituting (16) into equations (10), we get a rational fractional equations which is just for e^{ξ}. Collecting all the terms with the same power of e^{ξ} and letting their coefficients be zero yields a set of algebraic system for $a,\ b,\ c,\ f,\ g,\ h,\ o,\ p,\ q,\ r,\ s,\ w,\ k,\ \alpha,\ \beta,\ \omega$. Using Maple, package, we can deal with the above algebra equations and output directly the required solution. Further, substituting the solution and $\eta = \alpha x + \beta t,\ \xi = kx + \omega t$ into (15) and (8), we obtain the following exact solution for equation (7).

Case 1. $q = -k^2 r,\ b = ifkm,\ f = f,\ a = h = c = o = p = 0,\ w = \dfrac{rg^2}{f^2},\ \beta = \pm 1,\ s = \dfrac{2gr}{f},\ \alpha = -1,\ \omega = k,\ m = m,\ r = r,\ k = k,\ g = g,\ \psi_{10}(x,t) = \dfrac{ie^{i(-x\pm t)} fkm}{fe^{(kx+kt)} + g},\ \phi_{10}(x,t) = \dfrac{-k^2 g^2}{\left(ge^{(kx-kt)} + h\right)^2}$.

Case 2. $q = -k^2 r,\ b = ifkm,\ f = f,\ a = s = w = h = o = p = 0,\ \beta = \pm 1,\ c = ikmg,\ \alpha = -1,\ \omega = k,\ m = m,\ r = r,\ k = k,\ g = g,\ \psi_{11}(x,t) = \dfrac{ie^{i(-x\pm t)} km}{fe^{(kx+kt)} + g},\ \phi_{11}(x,t) = \dfrac{-k^2}{e^{2(kx+kt)}}$.

Remark 4.1. We claim that the tanh method can be extended by replacing tanh function with some generalized functions $f(x)$, such as polynomial function, trigonometric function and Jacobi elliptical function. As an example, we obtain three classes exact solutions for KGS equation.

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The adjacent vertex distinguishing I-total chromatic number of ladder graph

Yang Suiyi †, He Wansheng ‡ and Ren Shengzhang ♯

Department of Mathematics,
Tianshui Normal University, Gansu Tianshui−741 001, China
E-mail: yangzhangyike@yahoo.com.cn

Abstract The I-total coloring of a graph G is an assignment of some colors to its vertices and edges such that no two adjacent vertices receive the same color and no two adjacent edges receive the same color. Under an I-total coloring of G, the color set of a vertex x of G is the set of all colors which are assigned to vertex x or the edges incident to x. An I-total coloring is called adjacent vertex distinguishing if any two adjacent vertices have different color sets. The minimum number of colors required in an adjacent vertex-distinguishing I-total coloring is called adjacent vertex-distinguishing I-total chromatic number. The adjacent vertex-distinguishing I-total chromatic number of Ladder graph is discussed in this paper.

Keywords Ladder graph, adjacent vertex-distinguishing, I-total coloring, adjacent vertex-dist -inguishing, I-total chromatic number.

§1. Introduction and preliminaries

Coloring is a important research area of graph theory. Some new colorings of graphs are produced from applied areas of computer science, information science and light transmission, such as vertex distinguishing proper edge coloring [1], adjacent vertex distinguishing proper edge coloring [2] and adjacent vertex distinguishing total coloring [3,4] and so on, those problems are very difficult. Recently, Zhang et al. [5] propose the adjacent vertex-distinguishing I-total coloring of graphs. In this paper, we obtain the adjacent vertex-distinguishing I-total chromatic number of ladder graph by constructing method.

Definition 1.1. [5] A proper total k-coloring of G is a mapping $\pi : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ such that two adjacent elements of $V(G) \cup E(G)$ are assigned distinct colors, and the incident elements are dyed different colors. Let $C(u) = \{\pi(u)\} \cup \{\pi(uv)|uv \in E(G)\}$ for any vertex $u \in V(G)$. If $C(u) \neq C(v)$ for $uv \in E(G)$, we say that π is an adjacent-vertex distinguishing total k-coloring (a k-$AVDTC$) of G. The minimum number such that G admits a k-$AVITC$, denoted by $\chi_{at}(G)$, is called the adjacent-vertex distinguishing I-total chromatic number of G.

Conjecture 1.1. [5] For a simple graph G, then $\chi_{at}(G) \leq \Delta(G) + 3$.

Definition 1.2. [5] A total k-coloring of G is a mapping $\pi : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ such that two adjacent elements of $V(G) \cup E(G)$ are assigned distinct colors. Let $C(u) =$

$\{\pi(u)\} \cup \{\pi(uv)|uv \in E(G)\}$ for any vertex $u \in V(G)$. If $C(u) \neq C(v)$ for $uv \in E(G)$, we say that π is an adjacent-vertex distinguishing I-total k-coloring (a k-$AVDITC$) of G. The minimum number such that G admits a k-$AVDITC$, denoted by $\chi^i_{at}(G)$, is called the adjacent-vertex distinguishing I-total chromatic number of G.

Definition 1.3. [6] Suppose $G = (V, E)$ is a ladder graph such that:

$V(G) = \{u_i, v_i|i = 1, 2, \cdots, n\}$, $E(G) = \{u_1u_n\} \bigcup \{v_1v_n\} \bigcup \{u_iu_{i+1}|i = 1, 2, \cdots, n-1\} \bigcup \{v_iv_{i+1}|i = 1, 2, \cdots, n-1\} \bigcup \{u_iv_i|i = 1, 2, \cdots, n\} (n \geq 3)$.

Conjecture 1.2. [5] For a simple graph G, then $\chi^i_{at}(G) \leq \Delta(G) + 2$.

Undefined terminologies and notations follow [6,7].

§2. Riesz idempotent

Lemma 2.1. [5] For a simple graph G, there is

$$\chi^i_{at}(G) \geq \Delta.$$

If $\exists u,\ v \in E(G)$, and $d(u) = d(v) = \Delta$, then

$$\chi^i_{at}(G) \geq \Delta + 1.$$

§3. Main results

Theorem 2.1. Let G be a ladder graph, then

$$\chi^i_{at}(G) = 4.$$

Proof. According to the definition 1.2 of ladder graph, there is $\Delta(G) = 3$. Furthermore, we have $\chi^i_{at}(G) \geq 4$ through the lemma 2.1, in order to prove $\chi^i_{at}(G) = 4$, we only need to give a 4-$AVDIT$ coloring. At this moment, we define the coloring function τ as following:

$$\tau(u_i) = \begin{cases} 1, & i \equiv 1 \pmod 3; \\ 2, & i \equiv 2 \pmod 3; \\ 3, & i \equiv 0 \pmod 3). \end{cases} \quad i = 1,\ 2,\ \cdots,\ n-1.$$

$$\tau(u_iu_{i+1}) = \begin{cases} 1, & i \equiv 1 \pmod 3; \\ 2, & i \equiv 2 \pmod 3; \\ 3, & i \equiv 0 \pmod 3). \end{cases} \quad i = 1,\ 2,\ \cdots,\ n-1.$$

$$\tau(v_i) = \begin{cases} 2, & i \equiv 1 \pmod 3; \\ 3, & i \equiv 2 \pmod 3; \\ 1, & i \equiv 0 \pmod 3). \end{cases} \quad i = 1,\ 2,\ \cdots,\ n-1.$$

$$\tau(v_iv_{i+1}) = \begin{cases} 2, & i \equiv 1 \pmod 3; \\ 3, & i \equiv 2 \pmod 3; \\ 1, & i \equiv 0 \pmod 3). \end{cases} \quad i = 1,\ 2,\ \cdots,\ n-1.$$

$$\tau(u_n) = \begin{cases} 2, & n \not\equiv 0 \pmod 3; \\ 3, & n \equiv 0 \pmod 3). \end{cases}$$

$$\tau(v_n) = \begin{cases} 1, & n \not\equiv 1 \pmod 3; \\ 4, & n \equiv 1 \pmod 3). \end{cases}$$

$$\tau(u_1u_n) = \begin{cases} 3, & n \equiv 0 \pmod 3; \\ 4, & n \equiv 1 \pmod 3; \\ 2, & n \equiv 2 \pmod 3). \end{cases}$$

$$\tau(v_1v_n) = \begin{cases} 1, & n \equiv 0 \pmod 3; \\ 4, & n \equiv 1 \pmod 3; \\ 3, & n \equiv 2 \pmod 3). \end{cases}$$

The others are colored by 4.

Especially:

(1) When $n \equiv 1 \pmod 3$, we need to adjust τ as following:

$\tau(v_2) = 1,\ \tau(v_3) = 3,\ \tau(v_{n-2}) = 1,\ \tau(v_{n-1}) = 3,\ \tau(u_3) = 4,\ \tau(u_{n-1}) = 1$(when $n = 4$, $\tau(u_3) = \tau(u_{n-1}) = 1$), $\tau(u_1v_1) = 3,\ \tau(u_nv_n) = 2$.

(2) When $n \equiv 2 \pmod 3$, we need to adjust τ as following:

$\tau(u_1) = 3,\ \tau(u_3) = 4,\ \tau(v_2) = 1,\ \tau(v_3) = 3$.

And

$$C(u_1) = \begin{cases} \{1,3,4\}, & n \not\equiv 2 \pmod 3; \\ \{1,2,3,4\}, & n \equiv 2 \pmod 3). \end{cases}$$

$$C(u_i) = \begin{cases} \{1,3,4\}, & i \equiv 1 \pmod 3; \\ \{1,2,4\}, & i \equiv 2 \pmod 3; \\ \{2,3,4\}, & i \equiv 0 \pmod 3). \end{cases} \quad i = 2,\ 3,\ \cdots,\ n-1.$$

Besides, when $n \equiv 1 \pmod 3$, $C(u_{n-1}) = \{1,2,3,4\}$.

$$C(u_n) = \begin{cases} \{2,3,4\}, & n \not\equiv 2 \pmod 3; \\ \{1,2,4\}, & n \equiv 2 \pmod 3). \end{cases}$$

$$C(v_1) = \begin{cases} \{1,2,4\}, & n \equiv 0 \pmod 3; \\ \{2,3,4\}, & n \not\equiv 0 \pmod 3). \end{cases}$$

$$C(v_2)=\begin{cases}\{1,2,3,4\}, & n\not\equiv 0 \pmod 3;\\ \{2,3,4\}, & n\equiv 0 \pmod 3.\end{cases}$$

$$C(v_i)=\begin{cases}\{1,2,4\}, & i\equiv 1 \pmod 3;\\ \{2,3,4\}, & i\equiv 2 \pmod 3;\ \ i=3,\ 4,\ \cdots,\ n-1.\\ \{1,3,4\}, & i\equiv 0 \pmod 3.\end{cases}$$

Besides, when $n\equiv 1 \pmod 3$, $C(v_{n-2})=\{1,2,3,4\}$.

$$C(v_n)=\begin{cases}\{1,3,4\}, & n\equiv 0 \pmod 3;\\ \{1,2,4\}, & n\equiv 1 \pmod 3;\\ \{1,2,3,4\}, & n\equiv 2 \pmod 3.\end{cases}$$

It is obvious that no two adjacent vertices of sets are same, so τ is a 4-$AVDITC$ of G.

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A short interval result for the extension of the exponential divisor function

Li Yang

Department of Mathematics, Nanchang University
Nanchang, Jiangxi 330031, P. R. China
E-mail: yangli46@ncu.edu.cn

Abstract Let $\tau^{(e)}(n)$ denote the number of exponential divisor of n. Similar to the generalization from $d(n)$ to $d_k(n)$, we extended $\tau^{(e)}(n)$ to $(\tau_k^{(e)}(n))^{k-1}$. The aim of this paper is to establish a short interval result for the function $(\tau_3^{(e)}(n))^2$. This enriches the properties of the exponential divisor function.

Keywords The exponential divisor function, the generalized divisor function, short interval.
2000 Mathematics Subject Classification: 11E45

§1. Introduction

In 1972, M. V. Subbarao [3] established the definition of exponential divisor: Let $n > 1$ be an integer of canonical from $n = \Pi_{i=1}^{s} p_i^{a_i}$. The integer $d = \Pi_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of n if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$, notation: $d|_e n$. By convention $1|_e 1$. Besides, he also studied the mean value problem of exponential divisor function $\tau^{(e)}(n) = \sum_{d|_e n} 1$ and obtained:

$$\sum_{n \le x} \tau^{(e)}(n) = Ax + E(x),$$

where $E(x) = O(x^{\frac{1}{2}})$.

J. Wu [5] improved the result of M. V. Subbarao and obtained

$$\sum_{n \le x} \tau^{(e)}(n) = Ax + Bx^{\frac{1}{2}} + O(x^{\frac{2}{9}} \log x),$$

where

$$A = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{d(a) - d(a-1)}{p^a} \right),$$

$$B = \prod_p \left(1 + \sum_{a=5}^{\infty} \frac{d(a) - d(a-1) - d(a-2) + d(a-3)}{p^{a/2}} \right).$$

M. V. Subbarao [3] also proved that for any integers r, we have the estimate

$$\sum_{n \le x} (\tau^{(e)}(n))^r \sim A_r x,$$

where

$$A_r = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a}\right).$$

L. Tóth [1,2] proved

$$\sum_{n\le x} (\tau^{(e)}(n))^r = A_r x + x^{1/2} P_{2^r-2}(\log x) + O(x^{u_r+\epsilon}),$$

here $P_{2^r-2}(t)$ is a polynomial of degree $2^r - 2$ of t, $u_r = \frac{2^{r+1}-1}{2^{r+2}+1}$.

Similar to the generalization from $d(n)$ to $d_k(n)$, we extended $\tau^{(e)}(n)$ and established a definition as follows:

$$(\tau_k^{(e)}(n))^{k-1} = \left(\prod_{p_i^{a_i}\|n} d_k(a_i)\right)^{k-1}, \ k \ge 2.$$

Obviously $\tau_2^{(e)}(n) = \tau^{(e)}(n)$. In this paper we studied the case of $k = 3$ which means to study the properties of $(\tau_3^{(e)}(n))^2$ and obviously $(\tau_3^{(e)}(n))^2$ is a multiplicative function. The aim of this paper is to study the short interval case of $(\tau_3^{(e)}(n))^2$ and prove the following theorem.

Theorem 1.1. If $x^{\frac{1}{5}+2\epsilon} < y \le x$, then

$$\sum_{x<n\le x+y} (\tau_3^{(e)}(n))^2 = c_1 y + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}). \tag{1}$$

where $c_1 = Res_{s=1}F(s)$ and $F(s) := \sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^2}{n^s}$.

Notations. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant. We assume that $1 \le a \le b$ are fixed integers, and we denote by $d(a, b; k)$ the number of representations of k as $k = n_1^a n_2^b$, where n_1, n_2 are natural numbers, that is,

$$d(a, b; k) = \sum_{k=n_1^a n_2^b} 1,$$

and $d(a, b; k) \ll n^{\epsilon^2}$ will be used freely.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. Suppose $s = \sigma + it$ is a complex number ($\Re s > 1$), then

$$F(s) := \sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^2}{n^s} = \frac{\zeta(s)\zeta^2(2s)}{\zeta^3(5s)} G(s),$$

where the Dirichlet series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s = \sigma > \frac{1}{6}$.

Proof. Here $\tau_3^{(e)}(n)$ is multiplicative and by Euler product formula we have for $\sigma > 1$ that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\tau_3^{(e)}(n))^2}{n^s}\\
=&\prod_p\left(1+\frac{(\tau_3^{(e)}(p))^2}{p^s}+\frac{(\tau_3^{(e)}(p^2))^2}{p^{2s}}+\frac{(\tau_3^{(e)}(p^3))^2}{p^{3s}}+\frac{(\tau_3^{(e)}(p^4))^2}{p^{4s}}+\cdots\right)\\
=&\prod_p\left(1+\frac{(d_3(1))^2}{p^s}+\frac{(d_3(2))^2}{p^{2s}}+\frac{(d_3(3))^2}{p^{3s}}+\frac{(d_3(4))^2}{p^{4s}}+\frac{(d_3(5))^2}{p^{5s}}+\cdots\right)\\
=&\prod_p\left(1+\frac{1}{p^s}+\frac{9}{p^{2s}}+\frac{9}{p^{3s}}+\frac{36}{p^{4s}}+\frac{9}{p^{5s}}+\frac{81}{p^{6s}}+\cdots\right)\\
=&\zeta\prod_p\left(1+\frac{8}{p^{2s}}+\frac{27}{p^{4s}}-\frac{27}{p^{5s}}+\frac{72}{p^{6s}}+\cdots\right)\\
=&\zeta(s)\zeta^8(2s)\prod_p\left(1-\frac{9}{p^{4s}}-\frac{27}{p^{5s}}+\frac{24}{p^{6s}}+\cdots\right)\\
=&\frac{\zeta(s)\zeta^8(2s)}{\zeta^9(4s)\zeta^{27}(5s)}\prod_p\left(1+\frac{24}{p^{6s}}+\cdots\right)\\
=&\frac{\zeta(s)\zeta^8(2s)}{\zeta^9(4s)\zeta^{27}(5s)}G(s).
\end{aligned}\tag{2}$$

Now we write $G(s) := \sum_{n=1}^{\infty}\frac{g(n)}{n^s}$. It is easily seen the Dirichlet series is absolutely convergent for $\Re s = \sigma > \frac{1}{6}$.

Lemma 2.2. Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and

$$B(x,y;k,\epsilon) := \sum_{\substack{x<nm^k\leq x+y\\ m>x^\epsilon}} 1.$$

Then we have

$$B(x,y;k,\epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}}\log x. \tag{3}$$

Proof. This lemma is very important when studying the short interval distribution of 1-free number; see for example [4].

Let $a_1(n), a_2(n)$, $a_3(n)$ and $a_4(n)$ be arithmetic functions defined by the following Dirichlet series (for $\Re s > 1$):

$$\sum_{n=1}^{\infty}\frac{a_1(n)}{n^s} = \zeta(s)G(s). \tag{4}$$

$$\sum_{n=1}^{\infty}\frac{a_2(n)}{n^{2s}} = \zeta^8(2s). \tag{5}$$

$$\sum_{n=1}^{\infty}\frac{a_3(n)}{n^{4s}} = \zeta^{-9}(4s). \tag{6}$$

$$\sum_{n=1}^{\infty}\frac{a_4(n)}{n^{5s}}=\zeta^{-27}(5s). \tag{7}$$

Lemma 2.3. Let $a_1(n)$ be an arithmetic function defined by (4), then we have

$$\sum_{n\le x}a_1(n)=Cx+O(x^{\frac{1}{6}+\epsilon}), \tag{8}$$

where $C=Res_{s=1}\zeta(s)G(s)$.

Proof. Using lemma 1.1, it is easy to see that

$$\sum_{n\le x}|g(n)|\ll x^{\frac{1}{6}+\epsilon}.$$

Therefore from the definition of $g(n)$ and (4), it follows that

$$\begin{aligned}\sum_{n\le x}a_1(n) &= \sum_{mn\le x}g(n) && (9)\\ &= \sum_{n\le x}g(n)\sum_{m\le\frac{x}{n}}1 && (10)\\ &= \sum_{n\le x}g(n)(\frac{x}{n}+O(1)) && (11)\\ &= Cx+O(x^{\frac{1}{6}+\epsilon}), && (12)\end{aligned}$$

and $C=Res_{s=1}\zeta(s)G(s)$.

Next we prove our Theorem. From lemma 2.3 and the definition of $a_1(n), a_2(n)$, $a_3(n)$ and $a_4(n)$, we get

$$(\tau_3^{(e)}(n))^2=\sum_{n=n_1n_2^2n_3^4n_4^5}a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4),$$

and

$$a_1(n)\ll n^{\epsilon^2}, a_2(n)\ll n^{\epsilon^2}, a_3(n)\ll n^{\epsilon^2}, a_4(n)\ll n^{\epsilon^2}. \tag{13}$$

So we have

$$\begin{aligned}\sum_{n\le x+y}(\tau_3^{(e)}(n))^2-\sum_{n<x}(\tau_3^{(e)}(n))^2 &= \sum_{x<n_1n_2^2n_3^4n_4^5\le x+y}a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)\\ &= \sum_1+O(\sum_2+\sum_3+\sum_4), && (14)\end{aligned}$$

where

$$\begin{aligned}\sum_1 &= \sum_{\substack{n_2\le x^\epsilon\\ n_3\le x^\epsilon\\ n_4\le x^\epsilon}}a_2(n_2)a_3(n_3)a_4(n_4)\sum_{\frac{x}{n_2^2n_3^4n_4^5}<n_1\le\frac{x+y}{n_2^2n_3^4n_4^5}}a_1(n_1),\\ \sum_2 &= \sum_{\substack{x<n_1n_2^2n_3^4n_4^5\le x+y\\ n_2>x^\epsilon}}|a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)|,\\ \sum_3 &= \sum_{\substack{x<n_1n_2^2n_3^4n_4^5\le x+y\\ n_3>x^\epsilon}}|a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)|,\\ \sum_4 &= \sum_{\substack{x<n_1n_2^2n_3^4n_4^5\le x+y\\ n_4>x^\epsilon}}|a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)|. && (15)\end{aligned}$$

In view of lemma 3,

$$\begin{aligned}
\sum_1 &= \sum_{\substack{n_2\le x^\epsilon\\ n_3\le x^\epsilon\\ n_4\le x^\epsilon}} a_2(n_2)a_3(n_3)a_4(n_4)\left(\frac{Cy}{n_2^2n_3^4n_4^5}+O(\frac{x}{n_2^2n_3^4n_4^5})^{\frac16+\epsilon}\right)\\
&= c_1y+O\left(y\sum_{n_2>x^\epsilon}\frac{a_2(n_2)}{n_2^2}\sum_{n_3>x^\epsilon}\frac{a_3(n_3)}{n_3^4}\sum_{n_4>x^\epsilon}\frac{a_4(n_4)}{n_4^5}\right)\\
&+ O\left(x^{\frac16+\epsilon}\sum_{n_2\le x^\epsilon}\frac{a_2(n_2)}{n_2^{1/3+2\epsilon}}\sum_{n_3\le x^\epsilon}\frac{a_3(n_3)}{n_3^{2/3+4\epsilon}}\sum_{n_4\le x^\epsilon}\frac{a_4(n_4)}{n_4^{5/6+5\epsilon}}\right)\\
&= c_1y+O\left(yx^{-\frac{\epsilon}{6}}x^{-\frac{\epsilon}{6}}x^{-\frac{\epsilon}{6}}\right)+O\left(x^{\frac16+\epsilon}x^{\frac{\epsilon}{6}}x^{\frac{\epsilon}{6}}x^{\frac{\epsilon}{6}}\right)\\
&= c_1y+O(yx^{-\frac{\epsilon}{2}}+x^{\frac16+\frac32\epsilon}), \qquad (16)
\end{aligned}$$

where $c_1 = Res_{s=1}F(s)$.

$$\begin{aligned}
\sum_2 &\ll \sum_{\substack{x<n_1n_2^2n_3^4n_4^5\le x+y\\ n_2>x^\epsilon}} (n_1n_2n_3n_4)^{\epsilon^2}\\
&\ll x^{\epsilon^2}\sum_{\substack{x<n_1n_2^2n_3^4n_4^5\le x+y\\ n_2>x^\epsilon}} 1\\
&\ll x^{\epsilon^2}\sum_{\substack{x<m_1n_2^2n_4^5\le x+y\\ n_2>x^\epsilon}} d(1,4;m_1)\\
&\ll x^{2\epsilon^2}\sum_{\substack{x<m_1n_2^2n_4^5\le x+y\\ n_2>x^\epsilon}} 1\\
&\ll x^{2\epsilon^2}\sum_{\substack{x<mn_2^2\le x+y\\ n_2>x^\epsilon}} d(1,5;m)\\
&\ll x^{3\epsilon^2}B(x,y;2,\epsilon)\\
&\ll x^{3\epsilon^2}(yx^{-\epsilon}+x^{\frac15+\epsilon}\log x)\\
&\ll yx^{-\frac{\epsilon}{2}}+x^{\frac15+\frac32\epsilon}. \qquad (17)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\sum_3 \ll yx^{-\frac{\epsilon}{2}}+x^{\frac15+\frac32\epsilon},\\
&\sum_4 \ll yx^{-\frac{\epsilon}{2}}+x^{\frac15+\frac32\epsilon}. \qquad (18)
\end{aligned}$$

Now our theorem follows from (9)-(14).

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Terminal hosoya polynomial of thorn graphs

K. P. Narayankar †, S. B. Lokesh ‡, S. S. Shirkol ♯ and H. S. Ramane ♯

† Department of Mathematics, Mangalore University,
Mangalore - 574199, Karnataka, India
‡ Department of Mathematics, Shri Dharmasthala Manjunatheshwara College of
Engineering and Technology, Dharwad - 580002, Karnataka, India
♯ Department of Mathematics, Gogte Institute of Technology,
Belgaum - 590008, Karnataka, India
E-mail: kishori_pn@yahoo.co.in, shaila_shirkol@rediffmail.com,
hsramane@yahoo.com, sbloki83@gmail.com

Abstract The terminal Hosoya polynomial of thorn graphs is described. Also the terminal Hosoya polynomial for caterpillars, thorn stars, and thorn rings are obtained.

Keywords Terminal Hosoya polynomial, thorn graphs, thorn trees, thorn stars, thorn rings.
2000 Mathematics Subject Classification: 05C12

§1. Introduction

Let G be a connected graph with n vertices and m edges. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *degree* of a vertex v in G is the number of edges incident to it and is denoted by $deg_G(v)$. If $deg_G(v) = 1$ then v is called a *pendant vertex* or a *terminal vertex*. The *distance* between the vertices v_i and v_j in G is equal to the length of a shortest path joining them and is denoted by $d(v_i, v_j|G)$. The *diameter* of a graph G, denoted by $diam(G)$ is the maximum distance between any pair of vertices of G [2].

The Wiener index $W = W(G)$ of a graph G is defined as the sum of the distances between all pairs of vertices of G, that is

$$W = W(G) = \sum_{1 \le i < j \le n} d(v_i, v_j|G) \ .$$

This molecular structure descriptor was putforward by Harold Wiener [24] in 1947. Details on its chemical applications and mathematical properties can be found in [4, 10, 17, 23].

In 1988, Hosoya [13] introduced a graph polynomial $H(G, \lambda)$, defined as, if G is a connected graph with n vertices and m edges, and if $d(G, k)$ is the number of pairs of vertices of G that are at distance k, then

$$H(G, \lambda) = \sum_{k \ge 1} d(G, k)\lambda^k. \tag{1}$$

Hosoya called it the Wiener polynomial because $H'(G, 1) = W(G)$ where $H'(G, \lambda)$ denotes

the first derivative of $H(G, \lambda)$. But most contemporary authors call it the Hosoya polynomial in [6, 9, 14, 15, 18, 22].

Recently the terminal Wiener index $TW(G)$ was putforward by Gutman et al. [8]. The terminal Wiener index $TW(G)$ of a connected graph G is defined as the sum of the distances between all pairs of its pendant vertices. Thus if $V_T(G) = \{v_1, v_2, \ldots, v_k\}$ is the set of pendant vertices of G, then

$$TW(G) = \sum_{1\leq i<j\leq k} d(v_i, v_j|G).$$

For recent work on the terminal Wiener index, see [3, 7, 12, 16, 19].

In analogy of (1), we define the terminal Hosoya polynomial $TH(G, \lambda)$ of a graph G as

$$TH(G, \lambda) = \sum_{k\geq 1} d_T(G, k)\lambda^k,$$

where $d_T(G, k)$ is the number of pairs of pendant vertices of the graph G that are at distnce k.

It is easy to check that $TH'(G, 1) = TW(G)$, where $TH'(G, \lambda)$ is the first derivative of $TH(G, \lambda)$. In this paper we obtain the terminal Hosoya polynomial of thorn graphs.

§2. Terminal Hosoya polynomial of thorn graphs

Let G be a connected n-vertex graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and let $P = (p_1, p_2, \cdots, p_n)$ be an n-tuple of nonnegative integers. The thorn graph G_P is the graph obtained by attaching p_i pendant vertices to the vertex v_i of a graph G for $i = 1, 2, \cdots, n$. The p_i pendant vertices attached to the vertex v_i will be called the thorns of v_i.

The concept of thorny graph was introduced by Gutman[5] and eventually found a variety of applications in [1, 20, 21, 22].

Theorem 2.1. Let G be the connected graph with vertices $v_1, v_2, \cdots, v_n$. Let G_p be the thorn graph obtained by attaching p_i pendant vertices to the vertex v_i of G, $i = 1, 2, \cdots, n$. If $p_i > 0$ for all $i = 1, 2, \cdots, n$, then

$$TH(G_P, \lambda) = \sum_{i=1}^{n} \binom{p_i}{2}\lambda^2 + \sum_{1\leq i<j\leq n} p_i p_j \lambda^{2+d(v_i, v_j|G)}. \tag{2}$$

Proof. Consider p_i thorns attached to a vertex v_i, $i = i, 2, \cdots, n$. Each of these are at a distance two. Thus for each v_i, there are $\binom{p_i}{2}$ pairs of vertices which are at distance two. This leads to the first term of (2).

For the second term of (2), consider p_i thorns $v_1^i, v_2^i, \cdots, v_{p_i}^i$ attached to the vertex v_i and p_j thorns $v_1^j, v_2^j, \cdots, v_{p_j}^j$ attached to the vertex v_j of G, $i \neq j$. In G_p,

$$d(v_k^i, v_l^j|G_p) = 2 + d(v_i, v_j|G), \quad k = 1, 2, \cdots, p_i \text{ and } l = 1, 2, \cdots, p_j.$$

Since there are $p_i \times p_j$ pairs of thorns of such kind, their contribution to $TH(G_p, \lambda)$ is equal to $p_i p_j \lambda^{2+d(v_i, v_j|G)}$, $i \neq j$. This leads to the second term of (2).

(2) remains unchanged if some p_i's are equal to zero, provided that the corresponding vertices of the graph G are not pendant vertices.

Corollary 2.1. Let G be the connected graph with n vertices. If $p_i = p > 0$, $i = 1, 2, \cdots, n$, then

$$TH(G_p, \lambda) = \frac{np(p-1)}{2}\lambda^2 + p^2\lambda^2 \sum_{1 \le i < j \le n} \lambda^{d(v_i, v_j|G)}. \tag{3}$$

Corollary 2.2. Let G be the complete graph on n vertices. If $p_i = p > 0$, $i = 1, 2, \cdots, n$. Then

$$TH(G_P, \lambda) = \frac{np(p-1)}{2}\lambda^2 + \frac{p^2 n(n-1)}{2}\lambda^3.$$

Proof. If G is a complete graph then $d(v_i, v_j|G) = 1$ for all $v_i, v_j \in V(G)$, $i \neq j$. Therfore from (3)

$$\begin{aligned} TH(G_p, \lambda) &= \frac{np(p-1)}{2}\lambda^2 + p^2\lambda^2 \sum_{1 \le i < j \le n} \lambda \\ &= \frac{np(p-1)}{2}\lambda^2 + \frac{p^2 n(n-1)}{2}\lambda^3. \end{aligned}$$

Corollary 2.3. Let G be the connected graph with n vertices and m edges. If $diam(G) \le 2$ and $p_i = p > 0$, $i = 1, 2, \cdots, n$. Then

$$TH(G_P, \lambda) = \frac{np(p-1)}{2}\lambda^2 + mp^2\lambda^3 + \left(\frac{n(n-1)}{2} - m\right)p^2\lambda^4.$$

Proof. Since $diam(G) \le 2$, there are m pairs of vertices at distance 1 and $\binom{n}{2} - m$ pairs of vertices are at distance 2 in G. Therefore from (3)

$$\begin{aligned} TH(G_p, \lambda) &= \frac{np(p-1)}{2}\lambda^2 + p^2\lambda^2 \left(\sum_{m} \lambda + \sum_{\binom{n}{2} - m} \lambda^2 \right) \\ &= \frac{np(p-1)}{2}\lambda^2 + p^2\lambda^2 \left(m\lambda + \left(\binom{n}{2} - m \right) \lambda^2 \right) \\ &= \frac{np(p-1)}{2}\lambda^2 + mp^2\lambda^3 + \left(\frac{n(n-1)}{2} - m\right)p^2\lambda^4. \end{aligned}$$

Bonchev and Klein [1] proposed the terminology of thorn trees, where the parent graph is a tree. In a thorn tree if the parent graph is a path then it is a caterpillar [11].

Let P_{l+2} be the path on $l+2$ vertices, $l \ge 1$, labeled as $u_1, u_2, \cdots, u_l, u_{l+1}, u_{l+2}$, where u_i is adjacent to u_{i+1}, $i = 1, 2, \cdots, l+1$. Let $T = T(p_1, p_2, \cdots, p_l)$ be the thorn tree obtained from P_{l+2} by attaching $p_i \ge 0$ pendant vertices to u_{i+1}, $i = 1, 2, \cdots, l$.

Theorem 2.2. For a thorn tree $T = T(p_1, p_2, \cdots, p_l)$ of order $n \ge 3$, the terminal Hosoya polynomial is

$$TH(T, \lambda) = a_1\lambda + a_2\lambda^2 + \cdots + a_{l+1}\lambda^{l+1},$$

where

$$\begin{aligned} a_1 &= 0, \\ a_2 &= \sum_{i=1}^{l} \binom{p_i}{2} + p_1 + p_l, \\ a_k &= \sum_{i=1}^{l-k+2} p_i p_{i+k-2} + p_{k-1} + p_{l-k+2}, \qquad 3 \leq k \leq l, \\ a_{l+1} &= (p_1+1)(p_l+1). \end{aligned}$$

Proof. Let $A = \{u_1, u_2, \cdots, u_{l+1}, u_{l+2}\}$, $B_i = \{v_{i1}, v_{i2}, \cdots, v_{ip_i}\}$, $i = 1, 2, \cdots, l$ and $B = \bigcup_{i=1}^{l} B_i$.

There is no pair of pendant vertices which is at distance 1 in T. Therefore $a_1 = 0$.

Every pair of pendant vertices of B_i, $i = 1, 2, \cdots, l$ is at distance two. Also there are p_1 pairs of pendant vertices at distance two from the vertex u_1 and p_l pairs of pendant vertices at distance two from u_{l+2}. Therefore

$$a_2 = \sum_{i=1}^{l} \binom{p_i}{2} + p_1 + p_l.$$

For a_k, $3 \leq k \leq l$, $d(u, v|T) = k$ if $u \in B_i$ and $v \in B_{i+k-2}$, $i = 1, 2, \cdots, l$. There are $p_i \times p_{i+k-2}$ such pairs of pendant vertices which are at distance k. Also p_{k-1} pairs of pendant vertices are at distance k from u_1 and p_{l-k+2} pairs of pendant vertices are at distance k from u_{l+1}. Therefore

$$a_k = \sum_{i=1}^{l-k+2} p_i p_{i+k-2} + p_{k-1} + p_{l-k+2}, \qquad 3 \leq k \leq l.$$

Now $d(u, v) = l+1$ if $u \in B_1 \cup \{u_1\}$ and $v \in B_l \cup \{u_{l+2}\}$. Therefore $a_{l+1} = (p_1+1)(p_l+1)$.

Thorn star is a graph obtained by attaching pendant vertices to the vertices of star $K_{1,n}$ except to its central vertex.

Theorem 2.3. Let the thorn star $K^*_{1,n}$ is the graph obtained by attaching $p_i > 0$ pendant vertices to the i-th pendant vertex of a star $K_{1,n}$, $n \geq 2$, then

$$TH(K^*_{1,n}, \lambda) = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4,$$

where

$$a_1 = 0, \qquad a_2 = \sum_{i=1}^{n} \binom{p_i}{2}, \qquad a_3 = 0 \quad \text{and} \quad a_4 = \sum_{1 \leq i < j \leq n} p_i p_j.$$

Proof. Similar to the proof of theorem 2.2.

If C_n is the n-vertex cycle labeled cosecutively as $u_1, u_2, \cdots, u_n$, then the thorn ring C^*_n is obtained from C_n by attaching p_i pendant vertices to the vertex u_i, $i = 1, 2, \cdots, n$.

Theorem 2.4. For a thorn ring C^*_n, $n \geq 3$, the terminal Hosoya polynomial is

$$TH(C^*_n, \lambda) = a_1\lambda + a_2\lambda^2 + \cdots + a_{\lfloor n/2 \rfloor + 2}\lambda^{\lfloor n/2 \rfloor + 2}$$

where, if n is odd, then

$$a_1 = 0; \qquad a_2 = \sum_{i=1}^{n} \binom{p_i}{2};$$

and

$$a_k = \sum_{i=1}^{n-k+2} p_i p_{i+k-2} + \sum_{i=n-k+3}^{n} p_i p_{i-n+k-2}, \qquad 3 \leq k \leq \lfloor n/2 \rfloor + 2,$$

and if n is even, then

$$a_1 = 0; \qquad a_2 = \sum_{i=1}^{n} \binom{p_i}{2};$$

$$a_k = \sum_{i=1}^{n-k+2} p_i p_{i+k-2} + \sum_{i=n-k+3}^{n} p_i p_{i-n+k-2}, \qquad 3 \leq k \leq (n/2) + 1,$$

and

$$a_{(n/2)+2} = \sum_{i=1}^{n/2} p_i p_{i+(n/2)}.$$

Proof. The proof is analogous to that of theorem 2.2.

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Sigmoid function in the space of univalent function of Bazilevic type.

S. O. Olatunji [†], A. M. Gbolagade [‡], T. Anake [♯] and O. A. Fadipe-Joseph [♯]

† Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, P. M. B. 4000, Ogbomoso, Nigeria
‡ Department of Mathematics, Emmanuel Alayande College of Education, P. M. B. 1010, Oyo State, Nigeria
♯ Department of Mathematics, Covenant University, Ota, P. M. B. 1023, Ogun State, Nigeria
♯ Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, Nigeria
E-mail: olatfem_80@yahoo.com, gboladeam@yahoo.com, timothy.anake@convenantuniversity.edu.ng, famelov@unilorin.edu.ng

Abstract The authors investigated the coefficient bounds of the class $T_n^\alpha(\beta)$ as related to the class of modified Sigmoid activation functions.

Keywords analytic functions, coefficient bounds, Bazilevic and sigmoid functions.
2000 Mathematics Subject Classification: 30C45, Secondary 33E99

§1. Introduction

The theory of a special function has been in existence since nineteenth century due to the contributions of some reputable authors like Gauss, Jacobi, Franklein see [21] and some others. This function does not have a specific definition but it is a branch of Mathematics which is of utmost important to scientist and engineers who are concerned with Mathematical calculations. Special functions have a wide application in physics, Computer, engineering etc to mention just a few.

In twentieth century, the theory of special function has been overshadowed by other fields like real analysis, functional analysis, algebra, topology, differential equations. These functions also play a major role in geometric function theory.

Activation function is an information process that is inspired by the way biological nervous system such as brain, process information. It composed of large number of highly interconnected processing element (neurons) working together to solve a specific task. This function works in similar way the brain does, it learns by examples and can not be programmed to solve a specific task.

Activation function has a wide application in the real world bussiness problems. It can be used in industries to forecast sales and in identifying patterns trend. Also, it can be used

in modelling especially in human cardiovascular system, it can also be used in modelling of population. More so, it has application in, instant physician, architecture, electronic noses and etc.

Activation function can be categorized into three, namely, ramp function, threshold function and the sigmoid funtion. The most popular activation function is the sigmoid function because of its gradient descendent learning algorithm. Sigmoid function can be evaluated in different ways, it can be done by truncated series expansion, look-up tables or piecewise approximation.

This function is of the form $g(z) = \frac{1}{1+e^{-z}}$. It is useful because it is differentiable, which is important for weight learning algorithm. Neural networks can be used in complex learning task.

Sigmoid functions has the following features:

(i) It outputs real number between 0 and 1.

(ii) It map a large domain to a small range.

(iii) It never loses information because it is one-to-one function.

(iv) It increases monotonically.

The aforementioned features enable us to use sigmoid function in geometric function theory.

§2. Univalent functions

Univalent function is a function which does not take the same value twice, $f(z_1) \neq f(z_2)$ that is $z_1 \neq z_2\,(z_1, z_2 \in D)$ or otherwise. A function is said to be locally univalent at $z_0 \in D$, if it is univalent at the neighbourhood of z_0. Also, a function is analytic univalent if it is a conformal mapping. Hence, our concern is the class S of analytic and univalent function in the $U = \{z : |z| < 1\}$ normalized with $f(0) = 0$ and $f'(0) - 1 = 0$.

The fuction $g(z)$ has a Taylor series expansion of the form

$$g(z) = b_0 + b_1 z + b_2 z^2 + \cdots,$$

which gives

$$\frac{g(z) - b_0}{b_1} = z + \sum_{k=2}^{\infty} a_k z^k, \; b_1 \neq 0,$$

then

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Now, let γ be an analytic and univalent of the form (2.1) in the $U = \{z : |z| < 1\}$ normalized with $f(0) = 0$ and $f'(0) - 1 = 0$. Let us recall some definitions and concepts of classes of analytic functions. Let $f \in \gamma$. Then, $f \in S^*(\beta)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (0 \leq \beta < 1).$$

This class is called starlike class of analytic function, let $f \in \gamma$. Then, $f \in C(\beta)$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f(z)}\right\} > \beta \ \ (0 \leq \beta < 1)\,.$$

This class is called convex class of analytic function. Some well known authors like Bernardi [3], Darus and Ibrahim [5], Frasin [6], Goodman [7] etc to mention just but few have investigated the above two classes from different perspectives and they obtained many interesting results, the literatures on them littered everywhere.

Geometric function theory has wide applications in many physical problems, problems in physics, engineering etc.

Further more, Salagean [16] introduced the following differential operator:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^n f(z) &= D(D^{n-1} f(z)), \end{aligned}$$

and

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \tag{2}$$

where $f(z)$ is of the form (1).

Also, from (1), we can write that

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^\alpha. \tag{3}$$

Using binomial expansion on (3), we have

$$\begin{aligned} f(z)^\alpha \quad &= z^\alpha + \alpha a_2 z^{\alpha+1} + \left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2!} a_2^2\right) \\ &+ \left(\alpha a_4 + \frac{\alpha(\alpha-1)}{2!} 2a_2 a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} a_2^3\right) + \cdots . \end{aligned}$$

Thus

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}, \tag{4}$$

where $\alpha > 0$ (is real and it meant principal determinant only).

Applying differential operator (2) on (4), we obtained

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} (\alpha+k-1)^n a_k(\alpha) z^{\alpha+k-1}. \tag{5}$$

From (5), we have a subclass of γ to be $T_n^\alpha(\beta)$ defined by Opoola in [10],

$$Re\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta, \tag{6}$$

where ($\alpha > 0$ is real , $z \in U, 0 \leq \beta < 1, n = 0, 1, 2 \cdots$).

The study of analytic class in (6) started with the study of Bazilevic function that was discovered in 1955 by a Russian Mathematician called Bazilevic [2]. He defined the function by

$$f(z) = \left\{ \frac{\alpha}{1+\epsilon^2} \int_0^z \frac{P(v) - i\epsilon}{V(1 + \frac{1+\alpha\epsilon}{1+\epsilon^2})} g(v)^{\frac{\alpha}{1+\epsilon^2}} dv \right\}^{\frac{1+\epsilon^2}{\alpha}}, \tag{7}$$

where $p \in P$ and $g \in S^*$. The number $\alpha > 0$ and ϵ are real, and all powers meant principal determinant only. The family of function (7) is called Baziilevic function and it is denoted as $B(\alpha, \epsilon)$. The family of function $B(\alpha, \epsilon)$ breaks down to some well known subclasses of univalent functions.

For instance, if we take $\epsilon = 0$, we have

$$f(z) = \left\{ \alpha \int_0^z \frac{P(v)}{V} g(v)^\alpha dv \right\}^{\frac{1}{\alpha}}. \tag{8}$$

On differentiating (8), we have

$$\frac{zf' f(z)^{\alpha-1}}{g(z)^\alpha} = p(z). \tag{9}$$

Or equivalently

$$Re\left\{ \frac{zf' f(z)^{\alpha-1}}{g(z)^\alpha} \right\} > 0, \; z \in U.$$

The subclasses of Bazilevic functions satisfying (9) are called Bazilevic function of type α and denoted by $B(\alpha)$ see Singh [17].

In 1993, Noonman [9] gave a plausable description of functions of the class $B(\alpha)$ as those function in S for each $r < 1$, and the tangent to the curve $U_\alpha(r) = \left\{ \epsilon f \left(re^{i\theta} \right)^\alpha, 0 \leq \theta < 2\pi \right\}$ never turns back on itself as much as π radian. If $\alpha = 1$, the class $B(\alpha)$ reduces to the family of close-to-convex functions, that is

$$Re \frac{zf'(z)}{g(z)} > 0 \quad z \in U. \tag{10}$$

If we choose $g(z) = f(z)$ in (10), we have

$$Re \frac{zf'(z)}{f(z)} > 0, \; z \in U, \tag{11}$$

which implies that $f(z)$ is starlike. Furthermore, if we replace $f(z)$ by zf' in(11), we obtain

$$Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad z \in U, \tag{12}$$

which is the convexity condition. More over, if $g(z) = z$ in (9), then we have the family of $B_1(\alpha)$ see Singh [17] of functions satisfying

$$Re\left\{ \frac{zf' f(z)^{\alpha-1}}{z^\alpha} \right\} > 0, \; z \in U. \tag{13}$$

Many authors have studied this class of Bazilevic functions and they obtained many interesting results as contained in literatures (See Bernardi's Bibliography of Schlich function [4]).

In 1992, Abdulhalim [1] introduced a generalization of function satisfying (13) as

$$Re\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\} > 0, \tag{14}$$

where parameter α and D^n are as earlier defined. He denoted the class of function by $B_n(\alpha)$. It is easily seen that his generalization has extraneously included analytic function satisfying

$$Re\frac{f(z)^\alpha}{z^\alpha} > 0, \tag{15}$$

which are largely non univalent in the unit disk. By proving the inclusion $B_{n+1}(\alpha) \subset B_n(\alpha)$.

Abdulhalim in [1] showed that for all $n \in N$, each function of the class of $B_n(\alpha)$ is univalent in U.

Notable contributors like MacGregor [8], Noonman [9], Thomas [18], Tuan and Anh [19], Yamaguchi [20], Opoola [10], Oladipo [11], Oladipo and Breaz [12], Oladipo and Fadipe [13] had earlier considered various special cases of parameter n and α of (14) and established many properties of function in those particular cases.

Recently, Oladipo and Olatunji [14] used Cata's operator on this class of Bazilevic functions and some interesting results were obtained for coefficient bounds and the coefficient inequalities.

Let $p \in P$ be analytic and of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k,$$

with $p(0) = 1$ and $Rep(z) > 0$. This class of function is denoted by P known as Caratheodory functions.

The main aim of this work is to investigate how the sigmoid function is related to analytic and univalent functions of Bazilevic type in terms of coefficient bounds.

For the purpose of our result the following lemma shall be necessary.

Lemma 2.1. [15] Let $g(z)$ be a sigmoid function and $G(z) = 2g(z)$ then $G(z) \in P, |z| < 1$. Where $G(z)$ is a modified Sigmoid function.

Proof.

$$G(z) = \frac{2}{1+e^{-z}},$$

$$G(0) = 1,$$

$$G(z) + G(\overline{z}) = \frac{4(1 + Re(e^{-z}))}{(1+e^{-z})^2},$$

$$e^{-z} = e^{-x}\cos y - ie^{-x}\sin y,$$

$$Re(e^{-z}) = e^{-x}\cos y,$$

$$e^{-x} > 0.$$

This shows that $Re(e^{-z})$ depends only on the sign of $\cos y$. Since z is in the unit disc x and y vary from -1 and 1, since $x^2 + y^2 < 1$. Since $(-1, 1) \subset (\frac{-\pi}{2}, \frac{\pi}{2})$, $\cos y$ should be positive. Therefore, $Reg(z) > 0$. Hence, the function g under investigation maps the unit disk into the half plane with $g(0) = 1$ and is therefore in Caratheodory class.

The series form of a modified sigmoid function

$$G(z) = \frac{2}{1 + e^{-z}},$$

is given as

$$g(z) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n,$$

$$g(z) = \frac{1}{2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m\right],$$

$$2g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m,$$

hence

$$G(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m. \tag{16}$$

Here, we want to obtain the coefficient bound for the class $T_n^{\alpha}(\beta)$ as related to the modified activated sigmoid functions.

§3. Main result

Theorem 3.1. Let $f \in T_n^{\alpha}(\beta)$, then

$$|a_2(\alpha)| \leq \frac{\alpha^{n-1}(1-\beta)}{2(\alpha+1)^n}, \ \alpha > 0, \tag{17}$$

$$|a_3(\alpha)| \leq \frac{\alpha^{2n-2}(1-\alpha)(1-\beta)^2}{8(\alpha+1)^{2n}}, \ 0 < \alpha < 1, \tag{18}$$

and

$$|a_4(\alpha)| \leq \begin{cases} F_1 + F_2 + F_3, & 0 < \alpha < 1; \\ F_3, & 1 < \alpha < 2; \\ F_1, & \alpha \leq 1, \end{cases}$$

where

$$F_1=-\frac{\alpha^{n-1}(1-\beta)}{24(\alpha+3)^n}, F_2=\frac{\alpha^{3n-3}(\alpha-1)^2(1-\beta)^3}{16(\alpha+1)^{3n}}, F_3=-\frac{\alpha^{3n-3}(\alpha-1)(\alpha-2)(1-\beta)^3}{48(\alpha+1)^{3n}}.$$

Proof. Let $f\in T_n^{\alpha}(\beta)$, define

$$G(z)=\frac{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}-\beta}{1-\beta}. \tag{19}$$

Recall that

$$\begin{aligned} f(z)^\alpha \quad &= z^\alpha+\alpha a_2 z^{\alpha+1}+\left(\alpha a_3+\frac{\alpha(\alpha-1)}{2!}a_2^2\right)\\ &+\left(\alpha a_4+\frac{\alpha(\alpha-1)}{2!}2a_2a_3+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}a_2^3\right)+\cdots, \end{aligned}$$

hence

$$(1-\beta)G(z)=1+\sum_{k=2}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k(\alpha)z^{k-1}-\beta. \tag{20}$$

Equivalently as

$$\begin{aligned} &(1-\beta)\left[\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m\right]\\ &=\alpha_1\left(\frac{\alpha+1}{\alpha}\right)^n a_2(\alpha)z+(\alpha_1a_3+\alpha_2a_2^2)\left(\frac{\alpha+2}{\alpha}\right)^n z^2\\ &+(\alpha_1a_4+2\alpha_2a_2a_3+\alpha_3a_2^3)\left(\frac{\alpha+3}{\alpha}\right)^n z^3\\ &+(\alpha_1a_5+\alpha_2(2a_2a_4+a_3^2)+3\alpha_3a_2^2a_3+\alpha_4a_2^4)\left(\frac{\alpha+4}{\alpha}\right)^n z^4+\cdots. \end{aligned}$$

Comparing the coefficient of (20), then we have

$$a_2(\alpha)=\frac{\alpha^n(1-\beta)}{2\alpha(\alpha+1)^n},$$

$$a_3(\alpha)=-\frac{\alpha^{2n}(\alpha-1)(1-\beta)^2}{8\alpha^2(\alpha+1)^{2n}},$$

$$a_4(\alpha)=-\frac{\alpha^n(1-\beta)}{24\alpha(\alpha+3)^n}+\frac{\alpha^{3n}(\alpha-1)^2(1-\beta)^3}{16\alpha^3(\alpha+1)^{3n}}-\frac{\alpha^{3n}(\alpha-1)(\alpha-2)(1-\beta)^3}{48\alpha^3(\alpha+1)^{3n}},$$

and this completes the proof.

Theorem 3.2. Let $f\in T_n^{\alpha}(\beta)$, then

$$\left|a_2a_4-a_3^2\right|\le\left(\frac{\alpha^{2n}(1-\beta)^2}{16\alpha^2(\alpha+1)^n}\right)\left|-\frac{\alpha^n}{3(\alpha+3)^n}+\left(\frac{\alpha^{2n}(\alpha-1)(1-\beta)^2}{2\alpha^2(\alpha+1)^{3n}}\right)\left(\frac{\alpha+1}{6}\right)\right|.$$

Theorem 3.3. Let $f\in T_n^{\alpha}(\beta)$, then

$$\left|a_3-\lambda a_2^2\right|\le\left(\frac{\alpha^{2n}(1-\beta)^2}{4\alpha^2(\alpha+1)^{2n}}\right)\left|\frac{-(\alpha-1)-2\lambda}{2}\right|.$$

Conclusively, with special choices of the principal determinations involved, many bounds could be obtained.

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k*-paranormal composition operators

S. Panayappan †, N. Jayanthi ‡ and D. Sumathi ♯

Post Graduate and Research Department of Mathematics, Government
Arts College, Coimbatore-18, India
E-mail: Jayanthipadmanaban@yahoo.in

Abstract In this paper, we charaecterise k*-paranormal composition operators and k*-paranormal weighted composition operators and their adjoints in L^2 spaces.

Keywords k*-paranormal operators, composition operators, weighted composition operators, aluthge transformation.

§1. Introduction and Preliminaries

Let $B(H)$ be the Banach Algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator, we mean an element from $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. For $0 \leq p < 1$, an operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$, T is called hyponormal. If $p = \frac{1}{2}$, T is called semi-hyponormal. An operator T is called paranormal, if $\|Tx\|^2 \leq \|T^2x\| \|x\|$, for every $x \in H$. Composition operators on hyponormal operators are studied by Alan Lambert [1]. Paranormal composition operators are studied by T. Veluchamy and S. Panayappan [9].

Let (X, Σ, λ) be a sigma-finite measure space. The relation of being almost everywhere, denoted by a.e, is an equivalence relation in $L^2(X, \Sigma, \lambda)$ and this equivalence relation splits $L^2(X, \Sigma, \lambda)$ into equivalence classes. Let T be a measurable transformation from X into itself. $L^2(X, \Sigma, \lambda)$ is denoted as $L^2(\lambda)$. The equation $C_T f = f \circ T$, $f \in L^2(\lambda)$ defines a composition transformation on $L^2(\lambda)$. T induces a composition operator C_T on $L^2(\lambda)$ if (i) the measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ and (ii) the Radon-Nikodym derivative $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$ is essentially bounded(Nordgren). Harrington and Whitley have shown that if $C_T \in B(L^2(\lambda))$, then $C_T^* C_T f = f_0 f$ and $C_T C_T^* f = (f_0 \circ T)Pf$ for all $f \in L^2(\lambda)$ where P denotes the projection of $L^2(\lambda)$ onto $\overline{ran(C_T)}$. Thus it follows that C_T has dense range if and only if $C_T C_T^*$ is the operator of multiplication by $f_0 \circ T$, where f_0 denotes $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$.

Every essentially bounded complex valued measurable function f_0 induces a bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_0} f = f_0 f$, for every $f \in L^2(\lambda)$. Further $C_T^* C_T = M_{f_0}$ and ${C_T^*}^2 {C_T}^2 = M_{h_0}$. Let us denote $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$ by h i.e. f_0 by h and $\frac{\mathrm{d}(\lambda T^{-k})}{\mathrm{d}\lambda}$ by h_k, where k is a positive integer greater than or equal to one. Then $C_T^* C_T = M_h$ and ${C_T^*}^2 {C_T}^2 = M_{h_2}$. In general, ${C_T^*}^k {C_T}^k = M_{h_k}$, where M_{h_k} is the multiplication operator on $L^2(\lambda)$ induced by the complex valued measurable function h_k.

Definition 1.1. [10] An operator T is called k*- paranormal for a positive integer k, if for

every unit vector x in H, $\left\|T^k x\right\| \geq \|T^* x\|^k$.

Theorem 1.1. [10] For $k \geq 3$, there exists a k*-paranormal operator which is not k*-paranormal operator.

Theorem 1.2. An operator T is k*-paranormal for a positive integer k if and only if for any $\mu > 0$,

$$T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k \geq 0.$$

Proof. Let $\mu > 0$ and $x \in H$ with $\|x\| = 1$. Using arithmetic and geometric mean inequality, we get

$$\begin{aligned}
\frac{1}{k}\left\langle \mu^{-k+1}\left|T^k\right|^2 x, \quad x\right\rangle + \frac{k-1}{k}\langle \mu x, \quad x\rangle &= \left\langle \mu^{-k+1}\left|T^k\right|^2 x, \quad x\right\rangle^{\frac{1}{k}} \langle \mu x, \quad x\rangle^{\frac{k-1}{k}} \\
&= \left\|T^k x\right\|^{\frac{2}{k}} \\
&\geq \|T^* x\|^2 \\
&= \langle TT^* x, \quad x\rangle .
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\mu^{-k+1}}{k}\left\langle \left|T^k\right|^2 x, \quad x\right\rangle + \frac{(k-1)\mu}{k}\langle x, \quad x\rangle - \langle TT^* x, \quad x\rangle \geq 0. \\
\Rightarrow \quad & T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k \geq 0.
\end{aligned}$$

Conversely assume that $T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k \geq 0$.

If $\|T^* x\| = 0$, then k*- paranormality condition is trivially satisfied. If $x \in H$ with $\|T^* x\| \neq 0$ and $\|x\| = 1$, taking $\mu = \|T^* x\|^2$, we get

$$\left\|T^k x\right\| \geq \|T^* x\|^k .$$

Hence T is k*-paranormal.

In this paper we characterise k*-paranormal composition operators and k*-paranormal weighted composition operators.

§2. k*-paranormal composition operators

Theorem 2.1. For each positive integer k, $C_T \in B(L^2(\lambda))$ is k*-paranormal if and only if $h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu^k \geq 0$ a.e, for every $\mu > 0$.

Proof. By theorem 1.2, C_T is k*-paranormal if and only if

$$C_T^{*\,k} C_T^k - k\mu^{k-1} C_T C_T^* + (k-1)\mu^k I \geq 0, \text{ for every } \mu > 0.$$

This is true if and only if for every $f \in L^2(\lambda)$ and $\mu > 0$,

$$\langle M_{h_k} f, \, f\rangle - k\mu^{k-1}\left\langle M_{(h\circ T)P} f, \, f\right\rangle + (k-1)\mu^k \langle f, \, f\rangle \geq 0,$$

if and only if

$$\langle h_k f, \, f\rangle - k\mu^{k-1}\langle (h\circ T)P f, \, f\rangle + (k-1)\mu^k \langle f, \, f\rangle \geq 0,$$

if and only if

$$\langle h_k\chi_E,\, \chi_E\rangle - k\mu^{k-1}\left\langle (h\circ T)P\chi_E,\, \chi_E\right\rangle + (k-1)\mu^k\left\langle \chi_E,\, \chi_E\right\rangle \geq 0,$$

for every characteristic function χ_E of E in Σ, if and only if

$$\int_E (h_k - k\mu^{k-1}(h\circ T)P + (k-1)\mu^k)\,\mathrm{d}\lambda \geq 0 \text{ for every } E \text{ in } \Sigma,$$

if and only if

$$h_k - k\mu^{k-1}(h\circ T)P + (k-1)\mu^k \geq 0 \text{ a.e, for every } \mu > 0.$$

Corollary 2.1. For each positive integer k, $C_T \in B(L^2(\lambda))$ is k*-paranormal if and only if $h_k \geq (h^k\circ T)P$ a.e.

Theorem 2.2. For each positive integer k, C_T^* is k*-paranormal if and only if $(h_k\circ T^k)P_k \geq h^k$ a.e, where $P_i's$ are the projections of $L^2(\lambda)$ onto $\overline{ran(C_T^i)}$.

Proof. By theorem 1.2, C_T^* is k*-paranormal if and only if

$$\left\langle ({C_T}^k {C_T^*}^k - k\mu^{k-1}C_T^*C_T + (k-1)\mu^k)g,\, g\right\rangle \geq 0, \quad \text{for all}\, g\in L^2(\lambda),$$

if and only if $h_k\circ T^kP_k - k\mu^{k-1}h + (k-1)\mu^k \geq 0$ a.e, for all $\mu \geq 0$, if and only if $(h_k\circ T^k)P_k \geq h^k$ a.e.

Corollary 2.2. If $C_T \in B(L^2(\lambda))$ has dense range, then C_T^* is k*-paranormal if and only if $h_k\circ T^k \geq h^k$ a.e.

Corollary 2.3. If $C_T \in B(L^2(\lambda))$ has dense range, then both C_T and C_T^* are k*-paranormal if and only if $h_k\circ T^k \geq max(h^k, h^k\circ T^{k+1})$ a.e.

§3. Weighted composition operators and aluthge transformation of k*-paranormal operators

A weighted composition operator induced by T is defined as $Wf = w(f\circ T)$ where w is a complex valued Σ measurable function. Let w_k denote $w(w\circ T)(w\circ T^2)\cdots(w\circ T^{k-1})$. Then $W^kf = w_k(f\circ T)^k$. To examine the weighted composition operators effectively Alan Lambert [1] associated conditional expectation operator E with T as $E(\cdot/T^{-1}\Sigma) = E(\cdot)$. $E(f)$ is defined for each non-negative measurable function $f\in L^p(p\geq 1)$ and is uniquely determined by the conditions:

1. $E(f)$ is $T^{-1}\Sigma$ measurable.

2. if B is any $T^{-1}\Sigma$ measurable set for which $\int_B f\mathrm{d}\lambda$ converges, we have $\int_B f\mathrm{d}\lambda = \int_B E(f)\mathrm{d}\lambda$.

As an operator on L^p, E is the projection onto the closure of range of T and E is the identity operator on L^p if and only if $T^{-1}\Sigma = \Sigma$. Detailed discussion of E is found in [4], [5] and [7].

The following proposition due to Campbell and Jamison is well-known.

Theorem 3.1. [4] For $w \geq 0$,

1. $W^*Wf = h[E(w^2)] \circ T^{-1}f$.

2. $WW^*f = w(h \circ T)E(wf)$.

Since $W^k f = w_k(f \circ T^k)$ and $W^{*k} f = h_k E(w_k f) \circ T^{-k}$, we have $W^{*k}W^k = h_k E(w_k^2) \circ T^{-k} f$, for $f \in L^2(\lambda)$. Now we characterize k*-paranormal weighted composition operators.

Theorem 3.2. Let $W \in B(L^2(\lambda))$. Then W is k*-paranormal if and only if $h_k E(w_k^2) \circ T^{-k} - k\mu^{k-1} w(h \circ T)E(w) + (k-1)\mu^k \geq 0$ a.e, for every $\mu > 0$.

Proof. Since W is k*-paranormal,

$$W^{*k}W^k - k\mu^{k-1}WW^* + (k-1)\mu^k I \geq 0, \text{ for every } \mu > 0.$$

Hence

$$\int_E h_k E(w_k^2) \circ T^{-k} - k\mu^{k-1} w(h \circ T)E(w) + (k-1)\mu^k \mathrm{d}\,\lambda \geq 0,$$

for every $E \in \Sigma$ and so

$$h_k E(w_k^2) \circ T^{-k} - k\mu^{k-1} w(h \circ T)E(w) + (k-1)\mu^k \geq 0 \text{ a.e. for every } \mu > 0.$$

Corollary 3.1. Let $T^{-1}\Sigma = \Sigma$. Then W is k*-paranormal if and only if $h_k w_k^2 \circ T^{-k} - k\mu^{k-1} w^2(h \circ T) + (k-1)\mu^k \geq 0$ a.e, for every $\mu > 0$.

The Alugth transformation of T is the operator $\widetilde{T}$ given by $\widetilde{T} = |T|^{1/2} U |T|^{1/2}$ was introduced by Alugth [2]. More generally we may form the family of operators $T_r : 0 < r \leq 1$ where $T_r = |T|^r U |T|^{1-r}$[3]. For a composition operator C, the polar decomposition is given by $C = U|C|$ where $|C| = \sqrt{h} f$ and $Uf = \frac{1}{\sqrt{h \circ T}} f \circ T$. Lambert [5] has given a more general Alugth transformation for composition operators as $C_r = |C|^r U |C|^{1-r}$ as $C_r f = \left(\frac{h}{h \circ T}\right)^{r/2} f \circ T$. i.e. C_r is weighted composition with weight $\pi = \left(\frac{h}{h \circ T}\right)^{r/2}$.

Corollary 3.2. Let $C_r \in B(L^2(\lambda))$. Then C_r is of k*-paranormal if and only if $h_k E(\pi_k^2) \circ T^{-k} - k\mu^{k-1} \pi(h \circ T)E(\pi) + (k-1)\mu^k \geq 0$ a.e. for every $\mu > 0$.

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Difference cordial labeling of subdivided graphs

R. Ponraj †, S. Sathish Narayanan ‡ and R. Kala ♯

† Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, India
‡ Department of Mathematics, Thiruvalluvar College, Papanasam-627425, India
♯ Department of Mathematics, Manonmaniam Sundaranar University,
Tirunelveli-627012, India
E-mail: ponrajmaths@gmail.com, sathishrvss@gmail.com, karthipyi91@yahoo.co.in

Abstract Let G be a (p, q) graph. Let f be a map from $V(G)$ to $\{1, 2, \cdots, p\}$. For each edge xy, assign the label $|f(x) - f(y)|$. f is called a difference cordial labeling if f is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph. In this paper we investigate the difference cordial labeling behavior of subdivision of some graphs.

Keywords Subdivision, web, wheel, book, corona.
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§1. Introduction and preliminaries

Let $G = (V, E)$ be (p, q) graph. In this paper we have considered only simple and undirected graphs. The number of vertices of G is called the order of G and the number of edges of G is called the size G. The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uwv. The corona of G with H, $G \odot H$ is the graph obtained by taking one copy of G and p copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H. Graph labelling plays an important role of numerous fields of sciences and some of them are astronomy, coding theory, x-ray crystallography, radar, circuit design, communication network addressing, database management, secret sharing schemes etc. [3]. The graph labeling problem was introduced by Rosa. In 1967 Rosa introduced Graceful labeling of graphs [13]. In 1980, Cahit [1] introduced the concept of Cordial labeling of graphs. Riskin [12], Seoud and Abdel Maqusoud [14], Diab [2], Lee and Liu [5], Vaidya, Ghodasara, Srivastav, and Kaneria [15] were worked in cordial labeling. Ponraj et al. introduced k-Product cordial labeling [10], k-Total Product cordial labeling [11] recently. Motivated by the above work, R. Ponraj, S. Sathish Narayanan and R. Kala introduced difference cordial labeling of grpahs [6]. In [6, 7, 8, 9] difference cordial labeling behavior of several graphs like path, cycle, complete graph, complete bipartite graph, bistar, wheel, web, $B_m \odot K_1$, $T_n \odot K_2$ and some more standard

graphs have been investigated. In this paper, we investigate the difference cordial behaviour of $S(K_{1,n})$, $S(K_{2,n})$, $S(W_n)$, $S(P_n \odot K_1)$, $S(W(t,n))$, $S(B_m)$, $S(B_{m,n})$. Let x be any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms not defined here are used in the sense of Harary [4].

§2. Definition and properties

Definition 2.1. Let G be a (p,q) graph. Let $f : V(G) \rightarrow \{1, 2, \cdots, p\}$ be a bijection. For each edge uv, assign the label $|f(u) - f(v)|$. f is called a difference cordial labelling if f is $1-1$ and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labelled with 1 and not labelled with 1 respectively. A graph with a difference cordial labelling is called a difference cordial graph.

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Theorem 2.1. $S(K_{1,n})$ is difference cordial.

Proof. Let $V(S(K_{1,n})) = \{u\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{uu_i, u_iv_i : 1 \leq i \leq n\}$. Define a function $f : V(S(K_{1,n})) \rightarrow \{1, 2, \cdots, 2n+1\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, \quad 1 \leq i \leq n; \\ f(v_i) &= 2i, \quad\quad 1 \leq i \leq n, \end{aligned}$$

and $f(u) = 2n+1$. Since $e_f(0) = e_f(1) = n$, f is a difference cordial labelling of $S(K_{1,n})$.

A difference cordial labelling of $S(K_{1,8})$ is given in figure 1.

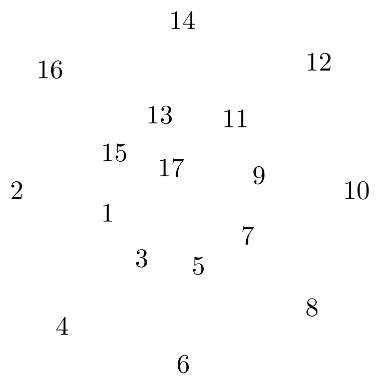

Figure 1

Theorem 2.2. $S(K_{2,n})$ is difference cordial.

Proof. Let $V(S(K_{2,n})) = \{u, v\} \cup \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(S(K_{2,n})) =$

$\{uv_i, v_iu_i, vw_i, w_iu_i : 1 \leq i \leq n\}$. Define $f : V(S(K_{2,n})) \rightarrow \{1,\ 2, \cdots, 3n+2\}$ by

$$\begin{aligned} f(u_i) &= 3i-1, & 1 \leq i \leq n; \\ f(v_i) &= 3i-2, & 1 \leq i \leq n; \\ f(w_i) &= 3i, & 1 \leq i \leq n. \end{aligned}$$

$f(u) = 3n+1$ and $f(v) = 3n+2$. Since $e_f(0) = e_f(1) = 2n$, f is a difference cordial labelling of $S(K_{2,n})$.

Theorem 2.3. $S(W_n)$ is difference cordial.

Proof. Let $V(S(W_n)) = \{u_i, w_i, v_i : 1 \leq i \leq n\} \cup \{u\}$ and $E(S(W_n)) = \left\{uv_i, v_iu_i, u_iw_i, w_iu_{i+1(\bmod\ n)} : 1 \leq i \leq n\right\}$. Define $f : V(S(W_n)) \rightarrow \{1,\ 2, \cdots, 3n+1\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, & 1 \leq i \leq n; \\ f(w_i) &= 2i, & 1 \leq i \leq n; \\ f(v_i) &= 2n+1+i, & 1 \leq i \leq n, \end{aligned}$$

and $f(u) = 2n+1$. Since $e_f(0) = e_f(1) = 2n$, $S(W_n)$ is difference cordial.

Subdivision of a Wheel $S(w_8)$ with a difference cordial labelling is shown in figure 2.

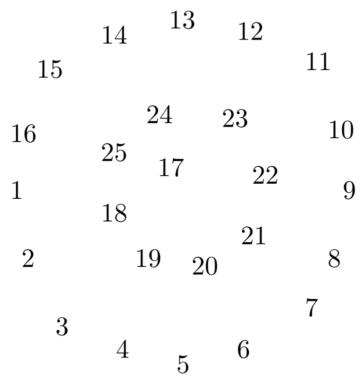

Figure 2

Theorem 2.4. Let G be a (p, q) graph and if $S(G)$ is difference cordial. Then $S(G \odot mK_1)$ is difference cordial.

Proof. Let f be a difference cordial labelling of $S(G)$. Let $V(S(G)) = \{u_i : 1 \leq i \leq n\}$, $V(S(G \odot mK_1)) = \left\{v_i^j, w_i^j : 1 \leq i \leq m, 1 \leq j \leq p\right\} \cup V(S(G))$ and $E(S(G \odot mK_1)) = \left\{u_iv_i^j, v_i^jw_i^j : 1 \leq i \leq m, 1 \leq j \leq p\right\} \cup E(S(G))$. Define a 1-1 map $g : V(S(G \odot mK_1)) \rightarrow \{1, 2, \cdots, (m+2)p\}$ by

$$\begin{aligned} g(u_i) &= f(u_i), & 1 \leq i \leq 2p; \\ f\left(v_i^j\right) &= 2p + (2j-2)m + 2i, & 1 \leq i \leq m,\ 1 \leq j \leq p; \\ f\left(w_i^j\right) &= 2p + (2j-2)m + 2i - 1, & 1 \leq i \leq m,\ 1 \leq j \leq p. \end{aligned}$$

Obviously the above labelling g is a difference cordial labelling of $S(G \odot mK_1)$.

Theorem 2.5. $S(P_n \odot K_1)$ is difference cordial.

Proof. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \le i \le n\}$ and let $V(S(P_n \odot K_1)) = V(P_n \odot K_1) \cup \left\{u_i^{'} : 1 \le i \le n-1\right\} \cup \left\{v_i^{'} : 1 \le i \le n\right\}$ and $E(S(P_n \odot K_1)) = \left\{u_i u_i^{'}, u_i^{'} u_{i+1} : 1 \le i \le n-1\right\} \cup \left\{u_i v_i^{'}, v_i^{'} v_i : 1 \le i \le n\right\}$. Define $f : V(S(P_n \odot K_1)) \to \{1, 2, \cdots, 4n-1\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, && 1 \le i \le n; \\ f\left(u_i^{'}\right) &= 2i, && 1 \le i \le n-1; \\ f\left(v_{n-i+1}^{'}\right) &= 2n-1+i, && 1 \le i \le n; \\ f\left(w_{n-i+1}^{'}\right) &= 3n-1+i, && 1 \le i \le n. \end{aligned}$$

Since $e_f(0) = e_f(1) = 2n-1$, $S(P_n \odot K_1)$ is difference cordial.

Theorem 2.6. $S(P_n \odot 2K_1)$ is difference cordial.

Proof. Let $V(S(P_n \odot 2K_1)) = \{u_i, v_i, w_i, x_i, y_i : 1 \le i \le n\} \cup \{z_i : 1 \le i \le n-1\}$ and $E(S(P_n \odot 2K_1)) = \{u_i v_i, v_i w_i, u_i x_i, x_i y_i : 1 \le i \le n\} \cup \{u_i z_i, z_i u_{i+1} : 1 \le i \le n-1\}$. Define $f : V(S(P_n \odot 2K_1)) \to \{1, 2, \cdots, 6n-1\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, && 1 \le i \le n; \\ f(z_i) &= 2i, && 1 \le i \le n-1; \\ f(v_{n-i+1}) &= 2n+2i-2, && 1 \le i \le n; \\ f(w_{n-i+1}) &= 2n+2i-1, && 1 \le i \le n; \\ f(x_i) &= 4n+i-1, && 1 \le i \le n; \\ f(y_i) &= 5n+i-1, && 1 \le i \le n. \end{aligned}$$

Since $e_f(0) = e_f(1) = 3n-1$, f is a difference cordial labelling of $S(P_n \odot 2K_1)$.

The Lotus inside a circle LC_n is a graph obtained from the cycle $C_n : u_1 u_2 \cdots u_n u_1$ and a star $K_{1,n}$ with central vertex v_0 and the end vertices $v_1 v_2 \cdots v_n$ by joining each v_i to u_i and $u_{i+1 \, (\text{mod } n)}$.

Theorem 2.7. $S(LC_n)$ is difference cordial.

Proof. Let the edges $u_i u_{i+1(\text{mod } n)}$, $v_0 v_i$, $v_i u_i$ and $v_i u_{i+1(\text{mod } n)}$ be subdivided by w_i, x_i, y_i and z_i respectively.

Define a map $f : V(S(LC_n)) \to \{1, 2, \cdots, 6n+1\}$ as follows:

$$\begin{aligned} f(u_i) &= 4i-3, && 1 \le i \le n; \\ f(v_i) &= 4i-1, && 1 \le i \le n; \\ f(w_i) &= 5n+1+i, && 1 \le i \le n; \\ f(x_i) &= 4n+1+i, && 1 \le i \le n; \\ f(y_i) &= 4i-2, && 1 \le i \le n; \\ f(z_i) &= 4i, && 1 \le i \le n, \end{aligned}$$

and $f(v_0) = 4n+1$. Obviously, the above vertex labelling is a difference cordial labelling of $S(LC_n)$.

The graph P_n^2 is obtained from the path P_n by adding edges that joins all vertices u and v with $d(u, v) = 2$.

Theorem 2.8. $S\left(P_n^2\right)$ is difference cordial.

Proof. Let $V\left(S\left(P_n^2\right)\right) = \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n-1\} \cup \{w_i : 1 \le i \le n-2\}$ and $E\left(S\left(P_n^2\right)\right) = \{u_iv_i, v_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_iw_i, w_iu_{i+2} : 1 \le i \le n-2\}$. Define $f : V\left(S\left(P_n^2\right)\right) \to \{1, 2, \cdots, 3n-3\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, && 1 \le i \le n-1; \\ f(v_i) &= 2i, && 1 \le i \le n-1; \\ f(w_i) &= 2n-1+i, && 2 \le i \le n-2, \end{aligned}$$

$f(u_n) = 2n$ and $f(w_1) = 2n-1$. Since $e_f(0) = e_f(1) = 2n-3$, f is a difference cordial labelling of $S\left(P_n^2\right)$.

Theorem 2.9. $S(K_2 + mK_1)$ is difference cordial.

Proof. Let $V(S(K_2 + mK_1)) = \{u, v, w, u_i, v_i, w_i : 1 \le i \le m\}$ and $E(S(K_2 + mK_1)) = \{uu_i, u_iw_i, w_iv_i, v_iv : 1 \le i \le m\} \cup \{uw, vw\}$. Define an injective map $f : V(S(K_2 + mK_1)) \to \{1, 2, \cdots, 3n+3\}$ as follows:

$$\begin{aligned} f(u_i) &= 3i+1, && 1 \le i \le m; \\ f(v_i) &= 3i+3, && 1 \le i \le m; \\ f(w_i) &= 3i+2, && 2 \le i \le m, \end{aligned}$$

$f(u) = 1$, $f(v) = 2$ and $f(w) = 3$. Since $e_f(0) = e_f(1) = 2m+1$, f is a difference cordial labelling of $S(K_2 + mK_1)$.

A difference cordial labelling of $S(K_2 + 4K_1)$ is given in figure 3.

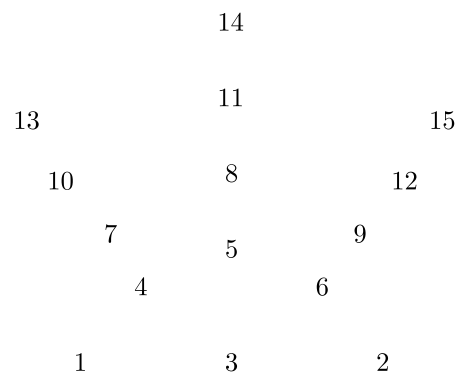

Figure 3

The sunflower graph S_n is obtained by taking a wheel with central vertex v_0 and the cycle $C_n : v_1v_2 \cdots v_nv_1$ and new vertices $w_1w_2 \cdots w_n$ where w_i is joined by vertices v_i, $v_{i+1 \pmod n}$.

Theorem 2.10. Subdivision of a sunflower graph $S(S_n)$ is difference cordial.

Proof. Let $V(S(S_n)) = \left\{u_i, u_i^{'}, v_i, w_i, w_i^{'}, x_i, u : 1 \le i \le n\right\}$ and $E(S(S_n)) = \left\{u_iu_i^{'}, u_i^{'}u_{i+1 (\bmod n)}, ux_i, x_iu_i, u_iw_i, w_iv_i, v_iw_i^{'}, w_i^{'}u_{i+1 (\bmod n)} : 1 \le i \le n\right\}$. Define f :

$V(S(S_n)) \rightarrow \{1, 2, \cdots, 6n+1\}$ by

$$\begin{aligned} f(u_i) &= 4i-3, & 1 \le i \le n; \\ f\left(u_i^{'}\right) &= 5n+1+i, & 1 \le i \le n; \\ f(w_i) &= 4i-2, & 1 \le i \le n; \\ f(v_i) &= 4i-1, & 1 \le i \le n; \\ f\left(w_i^{'}\right) &= 4i, & 1 \le i \le n; \\ f(x_i) &= 4n+1+i, & 1 \le i \le n, \end{aligned}$$

and $f(u) = 4n+1$. Since $e_f(0) = e_f(1) = 4n$, $S(S_n)$ is difference cordial.

The helm H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle. A flower Fl_n is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.

Theorem 2.11. Subdivision of graph of a flower graph $S(Fl_n)$ is difference cordial.

Proof. Let $V(S(Fl_n)) = \{u_i, v_i, w_i, x_i, y_i, z_i : 1 \le i \le n\} \cup \{u\}$ and $E(S(Fl_n)) = \left\{u_iv_i, v_iu_{i+1(\text{mod } n)}, ux_i, x_iu_i, u_iy_i, y_iw_i, w_iz_i, z_iu : 1 \le i \le n\right\}$. Define $f : V(S(Fl_n)) \rightarrow \{1, 2, \cdots, 6n+1\}$ by

$$\begin{aligned} f(u_i) &= 5i-1, & 1 \le i \le n; \\ f(v_i) &= 5n+i, & 1 \le i \le n; \\ f(w_i) &= 5i-3, & 1 \le i \le n; \\ f(x_i) &= 5i, & 1 \le i \le n; \\ f(y_i) &= 5i-2, & 1 \le i \le n; \\ f(z_i) &= 5i-4, & 1 \le i \le n, \end{aligned}$$

and $f(u) = 6n+1$. Since $e_f(0) = e_f(1) = 4n$, $S(Fl_n)$ is difference cordial.

The book B_m is the graph $S_m \times P_2$ where S_m is the star with $m+1$ vertices.

Theorem 2.12. $S(B_m)$ is difference cordial.

Proof. Let $V(S(B_m)) = \{u, v, w\} \cup \{u_i, v_i, w_i, x_i, y_i : 1 \le i \le m\}$ and $E(S(B_m)) = \{uw, vw\} \cup \{ux_i, x_iu_i, u_iw_i, w_iv_i, v_iy_i, y_iv : 1 \le i \le m\}$. Define a map $f : V(S(B_m)) \rightarrow \{1, 2, \cdots, 5m+3\}$ by

$$\begin{aligned} f(u_i) &= 4i, & 1 \le i \le m; \\ f(w_i) &= 4i+1, & 1 \le i \le m; \\ f(v_i) &= 4i+2, & 1 \le i \le m; \\ f(y_i) &= 4i+3, & 1 \le i \le m-1; \\ f(x_i) &= 4m+2+i, & 1 \le i \le m, \end{aligned}$$

$f(u) = 1$, $f(w) = 2$, $f(v) = 3$ and $f(y_m) = 5m+3$. Since $e_f(0) = e_f(1) = 3m+1$, $S(B_m)$ is difference cordial.

Prisms are graphs of the form $C_n \times P_n$.

Theorem 2.13. $S(C_n \times P_2)$ is difference cordial.

Proof. Let $V(S(C_n \times P_2)) = \{u_i, v_i, w_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(S(C_n \times P_2)) = \Big\{u_ix_i, x_iu_{i+1(\text{mod } n)}, u_iw_i, w_iv_i, v_iy_i, y_iv_{i+1(\text{mod } n)} : 1 \leq i \leq n\Big\}$. Define a function $f : V(S(C_n \times P_2)) \rightarrow \{1, 2, \cdots, 5n\}$ by

$$\begin{aligned} f(u_i) &= 2i-1, && 1 \leq i \leq n; \\ f(x_i) &= 2i, && 1 \leq i \leq n; \\ f(w_i) &= 2n+2i-1, && 1 \leq i \leq n; \\ f(v_i) &= 2n+2i, && 1 \leq i \leq n; \\ f(y_i) &= 4n+1+i, && 1 \leq i \leq n-1, \end{aligned}$$

and $f(y_n) = 4n+1$. Since $e_f(0) = e_f(1) = 3n$, f is a difference cordial labelling of $S(C_n \times P_2)$.

The graph obtained by joining the centers of two stars $K_{1,m}$ and $K_{1,n}$ with an edge is called Bistar $B_{m,n}$.

Theorem 2.14. $S(B_{m,n})$ is difference cordial.

Proof. Let $V(S(B_{m,n})) = \{u, v, w\} \cup \{u_i, x_i : 1 \leq i \leq m\} \cup \{v_i, y_i : 1 \leq i \leq n\}$ and $E(S(B_{m,n})) = \{uw, wv\} \cup \{ux_i, x_iu_i : 1 \leq i \leq m\} \cup \{vy_i, y_iv_i : 1 \leq i \leq n\}$. Define an injective map $f : V(S(B_{m,n})) \rightarrow \{1, 2, \cdots, 2m+2n+3\}$ by

$$\begin{aligned} f(u_i) &= 2i, && 1 \leq i \leq m; \\ f(v_i) &= 2m+2i, && 1 \leq i \leq n; \\ f(x_i) &= 2i-1, && 1 \leq i \leq m; \\ f(y_i) &= 2m+2i-1, && 1 \leq i \leq n. \end{aligned}$$

$f(u) = 2m+2n+3$, $f(v) = 2m+2n+2$ and $f(w) = 2m+2n+1$. Clearly the above vertex labelling is a difference cordial labelling of $S(B_{m,n})$.

The graph $K_2 \times K_2 \times \cdots \times K_2$ (n copies) is called a n-cube.

Theorem 2.15. Subdivision of a n-cube is difference cordial.

Proof. Let $G = K_2 \times K_2 \times \cdots \times K_2$ (n copies). Let the edges u_iu_{i+1}, v_iv_{i+1}, x_ix_{i+1}, w_iw_{i+1}, u_iv_i, v_ix_i, x_iw_i and w_iu_i be subdivided by $u_i^{'}$, $v_i^{'}$, $x_i^{'}$, $w_i^{'}$, y_i, $y_i^{'}$, z_i and $z_i^{'}$ respectively.

Define $f : V(S(G)) \to \{1, 2, \cdots, 12n-16\}$ by

$$\begin{aligned}
f(u_i) &= 5n+5i-14, & 3 \le i \le n-1;\\
f\left(u_i'\right) &= 5n+5i-10, & 2 \le i \le n-2;\\
f(v_i) &= 5n+5i-12, & 3 \le i \le n-1;\\
f\left(v_i'\right) &= 5n+5i-6, & 2 \le i \le n-2;\\
f(x_i) &= 5i-4, & 3 \le i \le n-1;\\
f\left(x_i'\right) &= 5i+2, & 2 \le i \le n-2;\\
f(w_i) &= 5i-6, & 3 \le i \le n-1;\\
f\left(w_i'\right) &= 5i-2, & 2 \le i \le n-2;\\
f(y_i) &= 5n+5i-13, & 3 \le i \le n-1;\\
f\left(y_i'\right) &= 10n-16+i, & 1 \le i \le n-1;\\
f(z_i) &= 5i-5, & 3 \le i \le n-1;\\
f\left(z_i'\right) &= 11n-17+i, & 1 \le i \le n-1.
\end{aligned}$$

$f(u_1) = 5n-7$, $f(u_2) = 5n-5$, $f\left(u_1'\right) = 5n-6$, $f(v_1) = 5n-1$, $f(v_2) = 5n-3$, $f\left(v_1'\right) = 5n-2$, $f(x_1) = 7$, $f(x_2) = 5$, $f\left(x_1'\right) = 6$, $f(w_1) = 1$, $f(w_2) = 3$, $f\left(w_1'\right) = 2$, $f(y_1) = 12n-17$, $f(y_2) = 5n-4$, $f(z_1) = 12n-16$ and $f(z_2) = 4$. Since $e_f(0) = e_f(1) = 8n-12$, f is a difference cordial labelling of $S(G)$. Jelly fish graphs $J(m,n)$ are obtained from a cycle $C_4 : v_1v_2v_3v_4v_1$ by joining v_1 and v_3 with an edge and appending m pendant edges to v_2 and n pendant edges to v_4.

Theorem 2.16. $S(J(m,n))$ is difference cordial.

Proof. Let the edges v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3 be subdivided by u, v, w, x, y respectively and the edges v_2u_i $(1 \le i \le m)$ and v_4w_j $(1 \le j \le n)$ be subdivided by u_i' and w_j' respectively. Define a one-to-one map $f : V(S(J(m,n))) \to \{1, 2, \cdots, 2m+2n+9\}$ by $f(v_1) = 2$, $f(v_2) = 7$, $f(v_3) = 5$, $f(v_4) = 8$, $f(u) = 3$, $f(v) = 6$, $f(w) = 4$, $f(x) = 1$, $f(y) = 9$,

$$\begin{aligned}
f(u_i) &= 2i+8, & 1 \le i \le m;\\
f\left(u_i'\right) &= 2i+9, & 1 \le i \le m;\\
f(w_i) &= 2m+2i+8, & 1 \le i \le n;\\
f\left(w_i'\right) &= 2m+2i+9, & 1 \le i \le n.
\end{aligned}$$

Clearly $e_f(0) = e_f(1) = m+n+5$. It follows that, f is a difference cordial labelling of $S(J(m,n))$. The helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle Cn. Koh et al. [3] Define a web graph. A web graph is obtained by joining the pendant points of the helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. Yang [3] has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendent point to the new cycle. $W(t,n)$ is the generalized web with t cycles C_n.

Theorem 2.17. $S(W(t,n))$ is difference cordial.

Proof. Let $C_n^{(i)}$ be the cycle $u_1^i, u_2^i \dots u_n^i u_1^i$. Let $V(W(t,n)) = \bigcup_{i=1}^{t} V\left(C_n^{(i)}\right) \cup \{z_i : 1 \le i \le n\}$ $\cup \{u\}$ and $V(S(W(t,n))) = \left\{v_i^j : 1 \le i \le n, 1 \le j \le t\right\} \cup \left\{w_i^j : 1 \le i \le n, 1 \le j \le t+1\right\} \cup$ $V(W(t,n))$ and $E(S(W(t,n))) = \left\{u_i^j v_i^j, v_i^j u_{i+1}^j : 1 \le i \le n-1, 1 \le j \le t\right\} \cup$ $\left\{u_n^j v_n^j, v_n^j u_1^j : 1 \le j \le t\right\} \cup \left\{u w_i^1 : 1 \le i \le n\right\} \cup \left\{w_i^j u_i^j, u_i^j w_{i+1}^j : 1 \le j \le t, 1 \le i \le n\right\} \cup$ $\left\{w_i^{t+1} z_i : 1 \le i \le n\right\}$. Define a map $f : V(S(W(t,n))) \to \{1, 2, \cdots, 3nt+2n+1\}$ as follows:

$$
\begin{aligned}
f\left(w_1^j\right) &= 2j-1, && 1 \le j \le t+1;\\
f\left(w_i^j\right) &= f\left(w_{i-1}^t\right)+2j, && 1 \le j \le t,\ 2 \le i \le n;\\
f\left(u_i^j\right) &= f\left(w_i^j\right)+1, && 1 \le j \le t,\ 2 \le i \le n;\\
f\left(z_i^j\right) &= f\left(w_i^{t+1}\right)+1, && 1 \le i \le n;\\
f\left(v_1^j\right) &= n(2t+2)+j, && 1 \le j \le t+1;\\
f\left(v_i^j\right) &= f\left(v_{i-1}^t\right)+j, && 1 \le j \le t+1,\ 2 \le i \le n,
\end{aligned}
$$

and $f(u) = 3nt+2n+1$. Since $e_f(0) = e_f(1) = n(2t+1)$, f is a difference cordial labelling of $S(W(t,n))$. A Young tableau $Y_{m,n}$ is a sub graph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting the vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a non increasing sequence. Let $V(P_m \times P_n) =$ $\{u_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ and $E(P_m \times P_n) = \{u_{i,j} ui, j+1 : 1 \le i \le m, 1 \le j \le n-1\} \cup$ $\{u_{i,j} u_{i+1,j} : 1 \le i \le m-1, 1 \le j \le n\}$.

Theorem 2.18. $S(Y_{n,n})$ is difference cordial.

Proof. Let $Y_{n,n}$ be a young tableau graph obtained from the grid $P_n \times P_n$. Let the edges $u_{i,j} u_{i+1,j}$ be subdivided by $w_{i,j}$. The order and size of $S(Y_{n,n})$ are $\frac{3n^2+5n-6}{2}$ and $2(n-1)(n+2)$ respectively. Define a map $f : V(S(Y_{n,n})) \to \left\{1, 2, \cdots, \frac{3n^2+5n-6}{2}\right\}$ as follows:

$$
\begin{aligned}
f(u_{1,j}) &= 2j-1, && 1 \le j \le n;\\
f(v_{1,j}) &= 2j, && 1 \le j \le n-1;\\
f(w_{1,j}) &= n^2-2n-2+j, && 1 \le j \le n.
\end{aligned}
$$

$$
\begin{aligned}
f(u_{2,j}) &= 2n+2j-2, && 1 \le j \le n;\\
f(v_{2,j}) &= 2n+2j-1, && 1 \le j \le n-1;\\
f(u_{i,j}) &= f(u_{i-1,n-i+3})+2j-1, && 3 \le i \le n,\ 1 \le j \le n-i+2;\\
f(v_{i,j}) &= f(u_{i,j})+1, && 3 \le i \le n,\ 1 \le j \le n-i+2;\\
f(w_{i,j}) &= f(w_{i-1,n-i+2})+j, && 2 \le i \le n-1,\ 1 \le j \le n-i+1.
\end{aligned}
$$

Since $e_f(0) = e_f(1) = (n-1)(n+2)$, f is a difference cordial labelling of $S(Y_{n,n})$.

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On right circulant matrices with trigonometric sequences

A. C. F. Bueno

Department of Mathematics and Physics, Central Luzon State University,
Science City of Muñoz 3120, Nueva Ecija, Philippines
Email: aldouz_cezar@yahoo.com

Abstract In this paper, the eigenvalues and the upper bounds for spectral norm and Euclidean norm of right circulant matrices with sine and cosine sequences were derived.

Keywords Circulant matrix, sine and cosine sequences.

§1. Introduction and preliminaries

In [1] and [2] the formulae for the determinant, eigenvalues, Euclidean norm, spectral norm and inverse of the right circulant matrices with arithmetic and geometric sequences were derived. The said right circulant matrices are as follows:

$$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & \cdots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \cdots & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & \cdots & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \cdots & a & ar \\ ar & ar^2 & ar^3 & \cdots & ar^{n-1} & a \end{pmatrix},$$

$$RCIRC_n(\vec{d}) = \begin{pmatrix} a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d \\ a+(n-1)d & ad & a+d & \cdots & a+(n-3)d & a+(n-2)d \\ a+(n-2)d & a+(n-1)d & a & \cdots & a+(n-4)d & a+(n-3)d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a+2d & a+3d & a+4d & \cdots & a & a+d \\ a+d & a+2d & a+3d & \cdots & a+(n-1)d & a \end{pmatrix},$$

with circulant vectors

$$\begin{aligned} \vec{g} &= (a, ar, ar^2, \cdots, ar^{n-2}, ar^{n-1}), \\ \vec{d} &= (a, a+d, a+2d, \cdots, a+(n-2)d, a+(n-1)d). \end{aligned}$$

The followings are basic identities on sine and cosine:

1. $\sin^2 x + \cos^2 x = 1$,

2. $\sin^2 2x = \frac{1-\cos^2 2x}{2}$,

3. $\cos^2 2x = \frac{1+\cos^2 2x}{2}$.
   The following are identities regarding $\sin kx$ and $\cos kx$ that will be used later:

4. $\sin kx = 2\cos x \sin{(k-1)x} - \sin{(k-2)x}$,

5. $\cos kx = 2\cos x \cos{(k-1)x} - \cos{(k-2)x}$,

6. $\sum_{k=0}^{n} \sin kx = \frac{\sin\left(\frac{1}{2}nx\right)\sin\left[\frac{1}{2}(n+1)x\right]}{\sin\left(\frac{1}{2}x\right)}$,

7. $\sum_{k=0}^{n} \cos kx = \frac{\cos\left(\frac{1}{2}nx\right)\sin\left[\frac{1}{2}(n+1)x\right]}{\sin\left(\frac{1}{2}x\right)}$.

Definition 1.1. Let the following be the circulant vectors of the right matrices $RCIRC_n(\vec{s})$, $RCIRC_n(\vec{c})$, $RCIRC_n(\vec{t})$ and $RCIRC_n(\vec{h})$:

$$\begin{aligned}
\vec{s} &= (0,\ \sin\theta,\ \sin 2\theta,\ \cdots, \sin{(n-1)\theta}), & (1)\\
\vec{c} &= (1,\ \cos\theta,\ \cos 2\theta,\ \cdots, \cos{(n-1)\theta}), & (2)\\
\vec{t} &= (0,\ \sin^2\theta,\ \sin^2 2\theta,\ \cdots, \sin^2{(n-1)\theta}), & (3)\\
\vec{h} &= (1,\ \cos^2\theta,\ \cos^2 2\theta,\ \cdots, \cos^2{(n-1)\theta}). & (4)
\end{aligned}$$

Definition 1.2. Let $\sigma_m,\ \gamma_m,\ \tau_m$ *and* δ_m be the eigenvalues of $RCIRC_n(\vec{s})$, $RCIRC_n(\vec{c})$, $RCIRC_n(\vec{t})$ and $RCIRC_n(\vec{h})$, respectively.

Definition 1.3. Let A be an $n \times n$ matrix then the Euclidean norm and spectral norm of A is denoted by $||A||_E$ and $||A||_2$, respectivley. These two are given by

$$\begin{aligned}
\|A\|_E &= \sqrt{\sum_{i,j=0}^{m,n} a_{ij}^2},\\
\|A\|_2 &= max\{|\lambda_m|\};\ \text{where } \lambda_m \text{ is an eigenvalue of } A.
\end{aligned}$$

Lemma 1.1.

$$\sum_{k=0}^{n-1} \omega^{-mk} = 0,\ \text{where } \omega = e^{2\pi i/n}.$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n-1} \omega^{-mk} &= \frac{1-\omega^{-mkn}}{1-\omega^{-mk}}\\
&= \frac{1-e^{2\pi ik}}{1-\omega^{-mk}}\\
&= 0.
\end{aligned}$$

§2. Main results

Theorem 2.1. The eigenvalues of $RCIRC_n(\vec{s})$ are

$$\sigma_0 = \frac{\sin\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)},$$
$$\sigma_m = \sum_{k=0}^{n-1}\left[2\cos\theta\sin(k-1)\theta-\sin(k-2)\theta\right]\omega^{-mk},$$

where $m = 1,\ 2,\ \cdots,\ n-1$ and $\omega = e^{2\pi i/n}$.

Proof. For $m = 0$,

$$\sigma_0 = \sum_{k=0}^{n-1}\sin k\theta = \frac{\sin\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}.$$

For $m \neq 0$,

$$\sigma_m = \sum_{k=0}^{n-1}\left[\sin k\theta\right]\omega^{-mk} = \sum_{k=0}^{n-1}\left[2\cos\theta\sin(k-1)\theta-\sin(k-2)\theta\right]\omega^{-mk}.$$

Theorem 2.2. The eigenvalues of $RCIRC_n(\vec{s})$ are the following:

$$\gamma_0 = \frac{\cos\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)},$$
$$\gamma_m = \sum_{k=0}^{n-1}\left[2\cos\theta\cos(k-1)\theta-\cos(k-2)\theta\right]\omega^{-mk}.$$

Proof. For $m = 0$,

$$\gamma_0 = \sum_{k=0}^{n-1}\cos k\theta = \frac{\cos\left(\frac{1}{2}n\theta\right)\sin\left[\frac{1}{2}(n+1)\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}.$$

For $m \neq 0$,

$$\gamma_m = \sum_{k=0}^{n-1}\left[\cos k\theta\right]\omega^{-mk} = \sum_{k=0}^{n-1}\left[2\cos\theta\cos(k-1)\theta-\cos(k-2)\theta\right]\omega^{-mk}.$$

Theorem 2.3. The eigenvalues of $RCIRC_n(\vec{t})$ are the following:

$$\begin{aligned}
\tau_0 &= \frac{n\sin\theta - 2\cos\left[(n-1)\theta\right]\sin n\theta}{2\sin\theta}, \\
\tau_m &= \sum_{k=0}^{n-1}\left\{\cos\left[2(k-2)\theta\right] - \cos 2\theta\cos\left[2(k-1)\theta\right]\right\}\omega^{-mk}.
\end{aligned}$$

Proof.

For $m = 0$,

$$\begin{aligned}
\tau_0 &= \sum_{k=0}^{n-1}\sin^2 k\theta \\
&= \sum_{k=0}^{n-1}\frac{1-\cos 2k\theta}{2} \\
&= \frac{n}{2} - \sum_{k=0}^{n-1}\cos 2k\theta \\
&= \frac{n\sin\theta - 2\cos\left[(n-1)\theta\right]\sin n\theta}{2\sin\theta}.
\end{aligned}$$

For $m \neq 0$,

$$\begin{aligned}
\tau_m &= \sum_{k=0}^{n-1}\left[\sin^2 k\theta\right]\omega^{-mk} \\
&= \sum_{k=0}^{n-1}\frac{1-\cos 2k\theta}{2}\omega^{-mk} \\
&= \frac{1}{2}\sum_{k=0}^{n-1}\omega^{-mk} - \sum_{k=0}^{n-1}\left[\cos 2k\theta\right]\omega^{-mk} \\
&= \sum_{k=0}^{n-1}\left\{\cos\left[2(k-2)\theta\right] - \cos 2\theta\cos\left[2(k-1)\theta\right]\right\}\omega^{-mk}.
\end{aligned}$$

via lemma 1.4 and identity 5.

Theorem 2.4. The eigenvalues of $RCIRC_n(\vec{h})$ are the following:

$$\begin{aligned}
\delta_0 &= \frac{n\sin\theta + 2\cos\left[(n-1)\theta\right]\sin n\theta}{2\sin\theta}, \\
\delta_m &= \sum_{k=0}^{n-1}\left\{\cos 2\theta\cos\left[2(k-1)\theta\right] - \cos\left[2(k-2)\theta\right]\right\}\omega^{-mk}.
\end{aligned}$$

For $m = 0$,

$$\begin{aligned}
\tau_0 &= \sum_{k=0}^{n-1} \cos^2 k\theta \\
&= \sum_{k=0}^{n-1} \frac{1+\cos 2k\theta}{2} \\
&= \frac{n}{2} + \sum_{k=0}^{n-1} \cos 2k\theta \\
&= \frac{n\sin\theta + 2\cos\left[(n-1)\theta\right]\sin n\theta}{2\sin\theta}.
\end{aligned}$$

For $m \neq 0$,

$$\begin{aligned}
\tau_m &= \sum_{k=0}^{n-1} \left[\cos^2 k\theta\right] \omega^{-mk} \\
&= \sum_{k=0}^{n-1} \frac{1+\cos 2k\theta}{2} \omega^{-mk} \\
&= \frac{1}{2}\sum_{k=0}^{n-1} \omega^{-mk} + \sum_{k=0}^{n-1} \left[\cos 2k\theta\right] \omega^{-mk} \\
&= \sum_{k=0}^{n-1} \left\{\cos 2\theta \cos\left[2(k-1)\theta\right] - \cos\left[2(k-2)\theta\right]\right\} \omega^{-mk}.
\end{aligned}$$

via lemma 1.4 and identity 5.

Theorem 2.5. $\|RCIRC_n(\vec{s})\|_2 \leq n-1$ and $\left\|RCIRC_n(\vec{t})\right\|_2 \leq n-1$.

Proof. Let $RCIRC_n(\vec{p}) = RCIRC_n(\vec{s})$ or $RCIRC_n(\vec{t})$ and $\mu_m = \sigma_m$ or τ_m.

$$\begin{aligned}
\|RCIRC_n(\vec{p})\|_2 &= max\left\{|\mu_m|\right\} \\
&= \left|\sum_{k=0}^{n-1} p_k \omega^{-mk}\right| \\
&\leq \sum_{k=0}^{n-1} |p_k|.
\end{aligned}$$

Note that $|\sin k\theta|, |\sin^2 k\theta| \in [0,1]$, so the theorem follows.

Theorem 2.6. $\|RCIRC_n(\vec{c})\|_2 \leq n$ and $\left\|RCIRC_n(\vec{h})\right\|_2 \leq n$.

Proof. Let $RCIRC_n(\vec{q}) = RCIRC_n(\vec{c})$ or $RCIRC_n(\vec{h})$ and $\phi_m = \gamma_m$ or δ_m.

$$\begin{aligned}
\|RCIRC_n(\vec{q})\|_2 &= max\left\{|\phi_m|\right\} \\
&= \left|\sum_{k=0}^{n-1} q_k \omega^{-mk}\right| \\
&\leq \sum_{k=0}^{n-1} |q_k|.
\end{aligned}$$

Note that $|\cos k\theta|, |\cos^2 k\theta| \in [0,1]$, so the theorem follows.

Theorem 2.7. $\|RCIRC_n(\vec{s})\|_E \leq \sqrt{n(n-1)}$ and $\left\|RCIRC_n(\vec{t})\right\|_E \leq \sqrt{n(n-1)}$.

Proof. Let $RCIRC_n(\vec{p}) = RCIRC_n(\vec{s})$ or $RCIRC_n(\vec{t})$,

$$\begin{aligned}\|RCIRC_n(\vec{p})\|_E &= \sqrt{n\sum_{k=0}^{n-1} p_k^2} \\ &\leq \sqrt{n\sum_{k=1}^{n-1} p_k^2} \\ &\leq \sqrt{n(n-1)}.\end{aligned}$$

Theorem 2.8. $\|RCIRC_n(\vec{c})\|_E \leq n$ and $\left\|RCIRC_n(\vec{h})\right\|_E \leq n$.

Proof. Let $RCIRC_n(\vec{q}) = RCIRC_n(\vec{c})$ or $RCIRC_n(\vec{h})$,

$$\begin{aligned}\|RCIRC_n(\vec{q})\|_E &= \sqrt{n\sum_{k=0}^{n-1} q_k^2} \\ &\leq \sqrt{n\sum_{k=0}^{n-1} 1} \\ &= \sqrt{n^2} \\ &= n.\end{aligned}$$

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Some theta identities and their implications[1]

Hai-Long Li † and Qian-Li Yang ‡

Department of Mathematics, Weinan Teachers University, Weinan City, Shaanxi Province.
E-mails: lihailong@wntc.edu.cn, darren2004@126.com

Abstract In this paper we establish an identity involving logarithmic derivative of theta function by the theory of elliptic functions. Using these identities we re-deduced some Ramanujan's modular identities from elementary approach, and also found many other new interesting identities.

Keywords Theta function, elliptic function, logarithmic derivative.
2000 Mathematics Subject Classification: Primary 26A48, 33B15; Secondary 26A51, 26D07, 26D10

§1. Introduction

Assume throughout this paper that $q = e^{\pi i \tau}$, where $\Im\tau > 0$. As usual, the classical Jacobi theta functions are defined as follows,

$$\begin{aligned}\theta_1(z|\tau) &= -iq^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)} e^{(2n+1)iz} \\ &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z, \quad (1)\\ \theta_2(z|\tau) &= q^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)} e^{(2n+1)iz} \\ &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \cos(2n+1)z, \quad (2)\\ \theta_3(z|\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nzi} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz, \quad (3)\\ \theta_4(z|\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nzi} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz. \quad (4)\end{aligned}$$

The $q-$ shifed factorial is defined by

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n),$$

[1]The author was supported partially by the Science Foundation of Shaanxi province

and sometimes is written as

$$(a_1, a_2, \cdots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)$$

With above notation, the celebrated Jacobi triple product identity can be expressed as follow

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n = (q; q)_\infty (z; q)_\infty (q/z; q)_\infty. \tag{5}$$

Employing the Jacobi triple product identity, we can derive the infinite product expressions for theta function.

Proposition 1.1(Infinite product representations for theta functions)

$$\theta_1(z|\tau) = 2q^{\frac{1}{4}} sinz (q^2; q^2)_\infty (q^2 e^{2iz}; q^2)_\infty (q^2 e^{-2iz}; q^2)_\infty, \tag{6}$$
$$\theta_2(z|\tau) = 2q^{\frac{1}{4}} (q^2; q^2)_\infty (-q^2 e^{2iz}; q^2)_\infty (-q^2 e^{-2iz}; q^2)_\infty, \tag{7}$$
$$\theta_3(z|\tau) = (q^2; q^2)_\infty (-q e^{2iz}; q^2)_\infty (-q e^{-2iz}; q^2)_\infty, \tag{8}$$
$$\theta_4(z|\tau) = (q^2; q^2)_\infty (q e^{2iz}; q^2)_\infty (q e^{-2iz}; q^2)_\infty. \tag{9}$$

When there is no confusion, we will use $\theta_i(z)$ for $\theta_i(z|\tau)$, θ_i' for $\theta_i'(z|\tau)$ to denote the partial derivative with respect to the variable z, and θ_i for $\theta_i(0|\tau), i = 1, 2, 3, 4$. From the above equations, the following facts are obvious

$$\begin{aligned}
\theta_1' &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)} = 2q^{\frac{1}{4}} (q^2; q^2)_\infty^3, \\
\theta_2 &= q^{\frac{1}{4}} \sum_{n=-\infty}^{\infty} q^{n(n+1)} = 2q^{\frac{1}{4}} (q^2; q^2)_\infty (-q^2; q^2)_\infty^2, \\
\theta_3 &= \sum_{n=-\infty}^{\infty} q^{n^2} = (q^2; q^2)_\infty (q; q^2)_\infty^2, \\
\theta_4 &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q^2; q^2)_\infty (-q; q^2)_\infty^2.
\end{aligned} \tag{10}$$

With respect to the (quasi) period π and $\pi\tau$, Jacobi theta functions $\theta_i, i = 1, 2, 3, 4$, satisfy the following relations

$$\begin{aligned}
&\theta_1(z+\pi|\tau) = -\theta_1(z|\tau), \quad \theta_1(z+\pi\tau|\tau) = -q^{-1} e^{-2iz} \theta_1(z|\tau), \\
&\theta_2(z+\pi|\tau) = -\theta_2(z|\tau), \quad \theta_2(z+\pi\tau|\tau) = q^{-1} e^{-2iz} \theta_2(z|\tau), \\
&\theta_3(z+\pi|\tau) = \theta_3(z|\tau), \quad \theta_3(z+\pi\tau|\tau) = q^{-1} e^{-2iz} \theta_3(z|\tau), \\
&\theta_4(z+\pi|\tau) = \theta_4(z|\tau), \quad \theta_4(z+\pi\tau|\tau) = -q^{-1} e^{-2iz} \theta_4(z|\tau).,
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\theta_1(z+\frac{\pi}{2}|\tau) &= \theta_2(z|\tau), \quad \theta_2(z+\frac{\pi}{2}|\tau) = -\theta_1(z|\tau),\\
\theta_3(z+\frac{\pi}{2}|\tau) &= \theta_4(z|\tau), \quad \theta_4(z+\frac{\pi}{2}|\tau) = \theta_3(z|\tau),\\
\theta_1(z+\frac{\pi\tau}{2}|\tau) &= iM\theta_4(z|\tau), \quad \theta_2(z+\frac{\pi\tau}{2}|\tau) = M\theta_3(z|\tau),\\
\theta_3(z+\frac{\pi\tau}{2}|\tau) &= M\theta_2(z|\tau), \quad \theta_4(z+\frac{\pi\tau}{2}|\tau) = iM\theta_1(z|\tau),\\
\theta_1(z+\frac{\pi\tau+\pi}{2}|\tau) &= M\theta_3(z|\tau), \quad \theta_2(z+\frac{\pi\tau+\pi}{2}|\tau) = -iM\theta_4(z|\tau),\\
\theta_3(z+\frac{\pi\tau+\pi}{2}|\tau) &= iM\theta_1(z|\tau), \quad \theta_4(z+\frac{\pi\tau+\pi}{2}|\tau) = M\theta_2(z|\tau),
\end{aligned} \tag{12}$$

where $M = q^{-1/4}e^{-iz}$.

The following trigonometric series expressions for the logarithmic derivative with respect to z of Jacobi Theta functions will be very useful in this paper,

$$\begin{aligned}
\frac{\theta_1'}{\theta_1}(z|\tau) &= \cot z + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2nz,\\
\frac{\theta_2'}{\theta_2}(z|\tau) &= -\tan z + 4\sum_{n=1}^{\infty}(-1)^n\frac{q^2n}{1-q^{2n}}\sin 2nz,\\
\frac{\theta_3'}{\theta_3}(z|\tau) &= 4\sum_{n=1}^{\infty}(-1)^n\frac{q^{n}}{1-q^{2n}}\sin 2nz,\\
\frac{\theta_4'}{\theta_4}(z|\tau) &= 4\sum_{n=1}^{\infty}\frac{q^{n}}{1-q^{2n}}\sin 2nz.
\end{aligned} \tag{13}$$

One may prove the Jacobi imaginary transformation formulas of the the theta functions employing the Poisson summation formula [**?**, pp.7-11]

Lemma 1.1. If z is complex, $\Im z > 0$, and $\sqrt{-i\tau} = 1$ for $\tau = i$, then we have

$$\begin{aligned}
\theta_1\left(\frac{z}{\tau}|-\frac{1}{\tau}\right) &= -i\sqrt{-i\tau}e^{iz^2/(\pi\tau)}\theta_1(z|\tau),\\
\theta_2\left(\frac{z}{\tau}|-\frac{1}{\tau}\right) &= \sqrt{-i\tau}e^{iz^2/(\pi\tau)}\theta_4(z|\tau),\\
\theta_3\left(\frac{z}{\tau}|-\frac{1}{\tau}\right) &= \sqrt{-i\tau}e^{iz^2/(\pi\tau)}\theta_3(z|\tau),\\
\theta_4\left(\frac{z}{\tau}|-\frac{1}{\tau}\right) &= \sqrt{-i\tau}e^{iz^2/(\pi\tau)}\theta_2(z|\tau).
\end{aligned} \tag{14}$$

In particular, by setting $z=0$ in above formulas we find that

$$\begin{aligned}
\theta_1'\Big(-\frac{1}{\tau}\Big) &= -i\tau\sqrt{-i\tau}\theta_1'(\tau)\\
\theta_2\Big(-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}\theta_4(\tau)\\
\theta_3\Big(-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}\theta_2(\tau)\\
\theta_4\Big(-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}\theta_3(\tau)
\end{aligned} \tag{15}$$

Suppose that $E_{2k}(\tau)$ is the normalized Eisenstein series of weight $2k$ defined as

$$E_{2k} = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^{2n}}{1-q^{2n}}$$

where B_{2k} is Bernoulli numbers defined as the coefficients in the power series

$$\frac{z}{e^z-1} = \sum_{k-0}^{\infty} B_k\frac{z^k}{k!}$$

And it is easy to show that $B_{2k+1}=0$ for $k\geq 1$, and the first few nonzero values of B_k are given by $B_0=1, B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42}$. Then near $z=0$, $\frac{\theta_1'}{\theta_1}(z|\tau)$ has the Laurent expansion formula

$$\frac{\theta_1'}{\theta_1}(z|\tau) = \sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}E_{2k}(\tau)z^{2k-1}.$$

Let $\sigma_k(n)$ denote the sum of the kth powers of divisors of n, namely, $\sigma_k(n)=\sum_{d|n}d^k$. Then the first three Eisenstein series E_{2k} are given by

$$\begin{aligned}
E_2 &= 1-24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\\
E_4 &= 1+240\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}\\
E_6 &= 1-504\sum_{n=1}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}}
\end{aligned} \tag{16}$$

Theorem 1.1. The sum of all the residues of an elliptic function in the period parallelogram is zero.

§2. Main theorem and proofs

Theorem 2.1. For x, y,

$$\frac{\theta_4'}{\theta_4}(x)+\frac{\theta_4'}{\theta_4}(y)-\frac{\theta_4'}{\theta_4}(x+y) = \theta_2\theta_3\frac{\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta_4(x)\theta_4(y)\theta_4(x+y)}. \tag{17}$$

Proof. We consider the following function

$$f(z) = \frac{\theta_4(z+x)\theta_4(z+y)\theta_4(z-x-y)}{\theta_1^2(z)\theta_4(z)},$$

by the definition of $\theta_i(z|\tau)$, we can readily verify that $f(z)$ is an elliptic function with periods π and $\pi\tau$. The only poles of $f(z)$ are 0 and $\frac{\pi\tau}{2}$. Furthermore, $\frac{\pi\tau}{2}$ is its simple pole and 0 is its pole with order two. By virtue of the residue theorem of elliptic functions, we have

$$Res(f; \frac{\pi\tau}{2}) + Res(f; 0) = 0. \tag{18}$$

And applying relation of θ_1 and θ_4 in (12) and L'Hopital' rules, we can obtain

$$\begin{aligned}
Res(f; \frac{\pi\tau}{2}) &= \lim_{z\to\frac{\pi\tau}{2}} (z - \frac{\pi\tau}{2}) f(z) \\
&= \lim_{z\to\frac{\pi\tau}{2}} \frac{(z-\frac{\pi\tau}{2})\theta_4(z+x)\theta_4(z+y)\theta_4(z-x-y)}{\theta_1^2(z)\theta_4(z)} \\
&= \frac{(iB)^3\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta_4'(\frac{\pi\tau}{2})\theta_1^2(\frac{\pi\tau}{2})} \\
&= -\frac{\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta_1'(0)\theta_4^2(0)}.
\end{aligned} \tag{19}$$

Next we compute $Res(f; 0)$,

$$\begin{aligned}
Res(f; 0) &= \lim_{z\to 0} \frac{d}{dz}(z^2 f(z)) \\
&= \lim_{z\to 0} z^2 f(z) \left[\frac{2}{z} + \frac{f'}{f}(z)\right] \\
&= \lim_{z\to 0} z^2 f(z) \left[\frac{2}{z} + \frac{\theta_4'}{\theta_4}(z+x) + \frac{\theta_4'}{\theta_4}(z+y) + \frac{\theta_4'}{\theta_4}(z-x-y) \right. \\
&\quad \left. - 2\frac{\theta_1'}{\theta_1}(z) - \frac{\theta_4'}{\theta_4}(z)\right] \\
&= \frac{\theta_4(x)\theta_4(y)\theta_4(x+y)}{\theta_1'^2(0)\theta_4(0)} \left[\frac{\theta_4'}{\theta_4}(x) + \frac{\theta_4'}{\theta_4}(y) - \frac{\theta_4'}{\theta_4}(x+y)\right].
\end{aligned} \tag{20}$$

From theorem 1.1, substituting (19) and (20) into (18), by performing a little reduction we can complete the proof of theorem.

Corollary 2.1.

$$\left(\frac{\theta_4'}{\theta_4}\right)'(y) - \left(\frac{\theta_4'}{\theta_4}\right)'(x) = (\theta_1')^2 \frac{\theta_1(x+y)\theta_1(x-y)}{\theta_4^2(x)\theta_4^2(y)}. \tag{21}$$

Proof. We differentiate (17) with respect to y, and then set $y = 0$, then

$$\left(\frac{\theta_4'}{\theta_4}\right)'(0) - \left(\frac{\theta_4'}{\theta_4}\right)'(x) = (\theta_2\theta_3)^2 \left(\frac{\theta_1(x)}{\theta_4(x)}\right)^2. \tag{22}$$

Now we combine with another elementary identity [7, p.467],

$$\theta_1^2(x)\theta_4^2(y) - \theta_1^2(y)\theta_4^2(x) = \theta_4^2\theta_1(x+y)\theta_1(x-y). \tag{23}$$

From (22) and (23), we can obtain

$$\begin{aligned}\left(\frac{\theta_4'}{\theta_4}\right)'(y)-\left(\frac{\theta_4'}{\theta_4}\right)'(x)&=(\theta_2\theta_3)^2\left[\left(\frac{\theta_1(x)}{\theta_4(x)}\right)^2-\left(\frac{\theta_1(y)}{\theta_4(y)}\right)^2\right]\\&=(\theta_2\theta_3)^2\frac{\theta_1^2(x)\theta_4^2(y)-\theta_1^2(y)\theta_4^2(x)}{\theta_4^2(x)\theta_4^2(y)}\\&=(\theta_1')^2\frac{\theta_1(x+y)\theta_1(x-y)}{\theta_4^2(x)\theta_4^2(y)}.\end{aligned}$$

This completes the proof of corollary 2.1.

Remark. The identity (21) is often written in terms of the Weierstrass elliptic and sigma functions as [7, p.451],

$$\wp(y)-\wp(x)=\frac{\sigma(x+y)\sigma(x-y)}{\sigma^2(x)\sigma^2(y)}.$$

Theorem 2.2. (Ramanujan's modular identity)

$$\sum_{n=0}^{\infty}\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{q^{5n+3}}{(1-q^{5n+3})^2}+\frac{q^{5n+4}}{(1-q^{5n+4})^2}=q\prod_{n=1}^{\infty}\frac{(1-5^{5n})^5}{1-q^n}. \tag{24}$$

Proof. We recall (13) for $\frac{\theta_4'}{\theta_4}(z|\tau)$,then

$$\left(\frac{\theta_4'}{\theta_4}\right)'(z|\tau)=8\sum_{n=1}^{\infty}\cos 2nz=4\sum_{n=1}^{\infty}\frac{nq^n(e^{2inz}+e^{-2inz})}{1-q^{2n}}, \tag{25}$$

in (21), we replace τ by $\frac{5}{2}\tau$, and choose $y=\frac{3\pi\tau}{4}$ and $x=\frac{\pi\tau}{4}$, using the Jacobi triple product identities for the theta functions. We can see that

$$\begin{aligned}\theta_4\left(\frac{\pi\tau}{4}|\frac{5\tau}{2}\right)\theta_4\left(\frac{3\pi\tau}{4}|\frac{5\tau}{2}\right)&=-q^{\frac14}\theta_1\left(\frac{\pi\tau}{2}|\frac{5\tau}{2}\right)\theta_1\left(\pi\tau|\frac{5\tau}{2}\right)\\\theta_1\left(\frac{\pi\tau}{2}|\frac{5\tau}{2}\right)\theta_1\left(\pi\tau|\frac{5\tau}{2}\right)&=-q^{-\frac14}\prod_{n=1}^{\infty}(1-q^{5n})(1-q^n)\\\theta_1'\left(0|\frac{5\tau}{2}\right)&=2q^{\frac58}\prod_{n=1}^{\infty}(1-q^{5n})^3.\end{aligned} \tag{26}$$

Then from (21), (25) and (26), we have

$$\begin{aligned}&4\sum_{n=0}^{\infty}\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{q^{5n+3}}{(1-q^{5n+3})^2}+\frac{q^{5n+4}}{(1-q^{5n+4})^2}\\&=4\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{5n}}-\frac{nq^{2n}}{1-q^{5n}}-\frac{nq^{3n}}{1-q^{5n}}+\frac{nq^{4n}}{1-q^{5n}}\\&=\left(\frac{\theta_4'}{\theta_4}\right)'\left(\frac{3\pi\tau}{4}|\frac{5\tau}{2}\right)-\left(\frac{\theta_4'}{\theta_4}\right)'\left(\frac{\pi\tau}{4}|\frac{5\tau}{2}\right)\\&=-\left(\theta_1'\left(0|\frac{5\tau}{2}\right)\right)^2\frac{\theta_1\left(\pi\tau|\frac{5\tau}{2}\right)\theta_1\left(\frac{\pi\tau}{2}|\frac{5\tau}{2}\right)}{\theta_4^2\left(\frac{\pi\tau}{4}|\frac{5\tau}{2}\right)\theta_4^2\left(\frac{3\pi\tau}{4}|\frac{5\tau}{2}\right)}\\&=4q\prod_{n=1}^{\infty}\frac{(1-5^{5n})^5}{1-q^n}.\end{aligned}$$

This completes the proof of the theorem.

From above procedure, we can rewrite theorem as

$$\left(\frac{\theta_4'}{\theta_4}\right)'\left(\frac{3\pi\tau}{4}|\frac{5\tau}{2}\right)-\left(\frac{\theta_4'}{\theta_4}\right)'\left(\frac{\pi\tau}{4}|\frac{5\tau}{2}\right)=4q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{1-q^n} \tag{27}$$

From Jacobi imaginary transformation lemma 1.1, we have

$$\left(\frac{\theta_4'}{\theta_4}\right)'\left(\frac{z}{\tau}|-\frac{1}{\tau}\right)=\tau^2\left(\frac{\theta_2'}{\theta_2}\right)'(z|\tau)+\frac{2i\tau}{\pi} \tag{28}$$

Then we have the following corollary:

Corollary 2.2

$$\begin{aligned}1-5\sum_{n=0}^{\infty}&\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}-\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}} \\ &=\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}.\end{aligned} \tag{29}$$

Proof. We can replace τ by $-\frac{1}{\tau}$ in (27) and applying to (28), then (27) becomes

$$\frac{\sqrt{5}}{4}\left[\left(\frac{\theta_2'}{\theta_2}\right)'\left(\frac{\pi}{10}|\frac{2\tau}{5}\right)-\left(\frac{\theta_2'}{\theta_2}\right)'\left(\frac{3\pi}{10}|\frac{2\tau}{5}\right)\right]=\prod_{n=1}^{\infty}\frac{(1-q^{\frac{4n}{5}})^5}{1-q^{4n}},$$

then replace τ by $\frac{5\tau}{4}$, it becomes

$$\frac{\sqrt{5}}{4}\left[\left(\frac{\theta_2'}{\theta_2}\right)'\left(\frac{\pi}{10}|\frac{\tau}{2}\right)-\left(\frac{\theta_2'}{\theta_2}\right)'\left(\frac{3\pi}{10}|\frac{\tau}{2}\right)\right]=\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}, \tag{30}$$

From (13), we can obtain

$$\left(\frac{\theta_2'}{\theta_2}\right)'(z|\tau)=-\sec^2 z+\sum_{n=1}^{\infty}\frac{(-1)^n nq^{2n}\cos 2nz}{1-q^{2n}}. \tag{31}$$

Next we can deduced (29) from (30) and (31) directly.

Therefore, This complete the corollary 2.2.

It should be remark that Bailey [1] also derived (24) and (29) in a similar fashion. However the key difference between our approach is that he derived corollary 2.1 from 6Ψ6 which is much less elementary than (17). Moreover by recasting these identities (27) and (30) in terms of theta function identity , we can easily discover and deduce other companion identities from the basic properties of the theta functions.

Corollary 2.3.

$$\sum_{n=0}^{\infty}\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{q^{5n+3}}{(1-q^{5n+3})^2}+\frac{q^{5n+4}}{(1-q^{5n+4})^2}$$
$$=q\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n-1})^2(1-q^{5n})^3(1+q^{5n})^2,$$
$$\sum_{n=0}^{\infty}\frac{(5n+1)q^{5n+1}}{1-q^{10n+2}}-\frac{(5n+2)q^{5n+2}}{1-q^{10n+4}}-\frac{(5n+3)q^{5n+3}}{1-q^{10n+6}}+\frac{(5n+4)q^{5n+4}}{1-q^{10n+8}}$$
$$=q\prod_{n=1}^{\infty}(1+q^n)(1-q^n)^3(1-q^{5n})^3(1+q^{5n})^3.$$

Proof. We recall the identity (21), and replace x by $x+\frac{\pi}{2}$ and y by $y+\frac{\pi}{2}$, then we obtain

$$\left(\frac{\theta_3'}{\theta_3}\right)'(y)-\left(\frac{\theta_3'}{\theta_3}\right)'(x)=(\theta_1')^2\frac{\theta_1(x+y)\theta_1(x-y)}{\theta_3^2(x)\theta_3^2(y)}$$

Now replacing τ by $\frac{5}{2}\tau$, and choosing $y=\frac{3\pi\tau}{4}$ and $x=\frac{\pi\tau}{4}$, we can deduce the corollary.

§3. Implications for modular identity of Ramanujan

Clearly, the identity (21) can generate unlimited number of similar identities. Since the proofs are identical and straightforward, we only list a few without providing the details of proofs here.

$$4\sum_{n=0}^{\infty}\frac{q^{6n+1}}{(1-q^{6n+1})^2}-\frac{q^{6n+2}}{(1-q^{6n+2})^2}-\frac{q^{6n+4}}{(1-q^{6n+4})^2}+\frac{q^{6n+5}}{(1-q^{6n+5})^2} \tag{32}$$
$$=\left(\frac{\theta_4'}{\theta_4}\right)'(\pi\tau|3\tau)-\left(\frac{\theta_4'}{\theta_4}\right)'\left(\frac{\pi\tau}{2}|3\tau\right)$$
$$=-(\theta_1'(0|3\tau))^2\frac{\theta_1\left(\frac{3\pi\tau}{2}|3\tau\right)\theta_1\left(\frac{\pi\tau}{2}|3\tau\right)}{\theta_4^2\left(\frac{\pi\tau}{2}|3\tau\right)\theta_4^2\left(\pi\tau|3\tau\right)}$$
$$=4q\prod_{n=1}^{\infty}\frac{(1-q^{6n})^3(1-q^{3n})^3}{(1-q^{2n})(1-q^n)},$$

$$4\sum_{n=0}^{\infty}\frac{q^{6n+2}}{(1-q^{6n+2})^2}-\frac{2q^{6n+3}}{(1-q^{6n+3})^2}+\frac{q^{6n+4}}{(1-q^{6n+4})^2} \tag{33}$$
$$=\left(\frac{\theta_4'}{\theta_4}\right)'(\frac{\pi\tau}{2}|3\tau)-\left(\frac{\theta_4'}{\theta_4}\right)'(0|3\tau)$$
$$=-\theta_2^2(0|3\tau)\theta_3^2(0|3\tau)\left(\frac{\theta_1\left(\frac{\pi\tau}{2}|3\tau\right)}{\theta_4\left(\frac{\pi\tau}{2}|3\tau\right)}\right)^2$$
$$=4q^2\prod_{n=1}^{\infty}\frac{(1-q^{6n})^6(1+q^{3n})^6}{(1-q^{2n})^2(1+q^n)^2}$$
$$=\frac{1}{4}\frac{\theta_2^6\left(0|\frac{3\tau}{2}\right)}{\theta_2^2\left(0|\frac{\tau}{2}\right)},$$

$$
\begin{aligned}
&4\sum_{n=0}^{\infty}\frac{q^{6n+1}}{(1-q^{6n+1})^2}-\frac{2q^{6n+3}}{(1-q^{6n+3})^2}+\frac{q^{6n+5}}{(1-q^{6n+5})^2} \\
&=\left(\frac{\theta_4'}{\theta_4}\right)'(3\pi\tau|3\tau)-\left(\frac{\theta_4'}{\theta_4}\right)'(0|3\tau) \\
&=-\theta_2^2(0|3\tau)\theta_3^2(0|3\tau)\frac{\theta_1^2(\pi\tau|3\tau)}{\theta_4^2(\pi\tau|3\tau)} \\
&=4q\prod_{n=1}^{\infty}\frac{(1-q^{6n})^2(1-q^{2n})^2}{(1-q^{2n-3})^2(1+q^{2n-1})^2} \\
&=\frac{1}{4}\theta_2^2\left(0|\frac{3\tau}{2}\right)\theta_2^2\left(0|\frac{\tau}{2}\right).
\end{aligned}
\tag{34}
$$

By acting the imaginary transformation on above three formulas (32), (33) and (34), we can obtain, respectively,

$$
\begin{aligned}
&1-3\sum_{n=0}^{\infty}\frac{(6n+1)q^{6n+1}}{1-q^{6n+1}}-\frac{2(6n+3)q^{6n+3}}{1-q^{6n+3}}+\frac{(6n+5)q^{6n+5}}{1-q^{6n+5}} \\
&=q\prod_{n=1}^{\infty}\frac{(1-q^n)^3(1-q^{2n})^3}{(1-q^{3n})(1-q^{6n})},
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&1-12\sum_{n=0}^{\infty}\frac{(6n+1)q^{6n+1}}{1-q^{6n+1}}-\frac{3(6n+2)q^{6n+2}}{1-q^{6n+2}}+\frac{4(6n+3)q^{6n+3}}{1-q^{6n+3}} \\
&\qquad-\frac{3(6n+4)q^{6n+4}}{1-q^{6n+4}}+\frac{(6n+5)q^{6n+5}}{1-q^{6n+5}} \\
&=q^2\prod_{n=1}^{\infty}\frac{(1-q^n)^6(1-q^{2n-1})^6}{(1-q^{3n})^2(1-q^{6n-3})^2} \\
&=\frac{\theta_4^6(0|\tau)}{\theta_4^2(0|3\tau)},
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&1-4\sum_{n=0}^{\infty}\frac{(6n+1)q^{6n+1}}{(1-q^{6n+1}}-\frac{(6n+2)q^{6n+2}}{1-q^{6n+2}}-\frac{(6n+4)q^{6n+4}}{1-q^{6n+4}}+\frac{(6n+5)q^{6n+5}}{1-q^{6n+5}} \\
&=q\prod_{n=1}^{\infty}\frac{(1-q^{3n})^2(1-q^n)^2}{(1+q^{3n})^2(1+q^n)^2} \\
&=\theta_4^2\left(0|\tau\right)\theta_4^2\left(0|3\tau\right).
\end{aligned}
\tag{37}
$$

Theorem 3.1. In [6], Ewell derived the following identity in virtue of the quintuple product identiy

$$
\frac{\theta_4^3(0|\tau)}{\theta_4(0|3\tau)}=1-6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1+q^{3n+1}}-\frac{q^{3n+2}}{1+q^{3n+1}}\right),
\tag{38}
$$

to study the sums of three squares. The identity (36) is the Lambert series for the square of (38)

On the other hand, in [5], authors define

$$b(q)=\prod_{n=1}^{\infty}\frac{(1-q^n)^3}{(1-q^{3n}} \quad \text{and} \quad c(q)=3q^{\frac{1}{3}}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3}{(1-q^n},$$

to study some cubic modular identities of Ramanujan. It is interesting to observe that the identities (32) and (35) are the Lambert series for $\frac{1}{9}c(q)c(q^2)$ and $b(q)b(q^2)$ respectively.

§4. Extension and generalization

It is surprising that hidden with (32), (33) and (34) is the following beautiful modular identity of Ramanujan,

$$\left(\frac{\theta_2(0|\frac{\tau}{3})}{\theta_2(0|3\tau)}-1\right)^3=\left(\frac{\theta_2(0|\tau)}{\theta_2(0|3\tau)}\right)^4-1. \tag{39}$$

To see this, we note that the identity (32) is exactly the difference of (33) and (34), therefore,

$$\frac{\theta_2^6(0|\frac{3\tau}{2})}{\theta_2^2(0|\frac{\tau}{2})}\left(\left(\frac{\theta_2(0|\frac{\tau}{2})}{\theta_2(0|\frac{3\tau}{2})}\right)^4-1\right)=16q\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3(1-q^{6n})^3}{(1-q^n)(1-q^{2n})}.$$

Hence

$$\begin{aligned}\left(\frac{\theta_2(0|\tau)}{\theta_2(0|\frac{3\tau}{2})}\right)^4-1&=\frac{1}{q}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3(1-q^{6n})^3(1-q^n)^2(1+q^n)^4}{(1-q^{3n})^6(1+q^{3n})^{12}(1-q^n)(1-q^{2n})} \\ &=\frac{1}{q}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3(1-q^{3n})^3(1+q^{3n})^3(1-q^n)^2(1+q^n)^4}{(1-q^{3n})^6(1+q^{3n})^{12}(1-q^n)(1-q^n)(1+q^n)} \\ &=\frac{1}{q}\prod_{n=1}^{\infty}\frac{(1+q^n)^3}{(1+q^{3n})^9}.\end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned}\theta_2(0|\tau)-\theta_2(0|9\tau)&=\sum_{n=-\infty}^{\infty}\left(q^{(n+\frac{1}{2})^2}-q^{(3n+\frac{3}{2})^2}\right)=\sum_{n=-\infty}^{\infty}\left(q^{(n+\frac{3}{2})^2}-q^{(3n+\frac{3}{2})^2}\right) \\ &=\sum_{n=-\infty}^{\infty}\left(q^{(3n+1+\frac{3}{2})^2}+q^{(3n-1+\frac{3}{2})^2}\right) \\ &=\sum_{n=-\infty}^{\infty}\left(q^{(3n-\frac{1}{2})^2}+q^{(3n+\frac{1}{2})^2}\right)=q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}q^{9n^2}\left(q^{3n}+q^{-3n}\right) \\ &=2q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty}q^{9n^2}q^{3n}=2q^{\frac{1}{4}}\theta_3\left(\frac{3\pi\tau}{2}|9\tau\right) \\ &=2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{18n})(1+q^{18n-6})(1+q^{18n-12}).\end{aligned}$$

Therefore,

$$\frac{\theta_2(0|\frac{\tau}{3})}{\theta_2(0|3\tau)}-1=\frac{\theta_3(0|\frac{\tau}{3})-\theta_2(0|3\tau)}{\theta_2(0|3\tau)}=q^{-2/3}\prod_{n=1}^{\infty}\frac{1+q^{2n}}{(1+q^{6n})^3}. \tag{41}$$

Clearly, the identity (39) now follows from (40) and (41).

We note that if we apply the imaginary transformation to (39) , then we obtain

$$\left(3\frac{\theta_4(0|\tau)}{\theta_4(0|\tau)}-1\right)^3=9\left(3\frac{\theta_4(0|3\tau)}{\theta_4(0|\tau)}\right)^4-1. \tag{42}$$

And using the familiar facts that $\theta_4(0|\tau+1)=\theta_3(0|\tau)$, $\theta_3(0|-\frac{1}{\tau})=\sqrt{-i\tau}\theta_3(0|\tau)$ and $\theta_4(0|-\frac{1}{\tau})=\sqrt{-i\tau}\theta_2(0|\tau)$, it is easy to derive the general formula

$$\left(\frac{\theta_i(0|\tau)}{\theta_i(0|9\tau)}-1\right)^3=\frac{\theta_i^4(0|3\tau)}{\theta_i^4(0|9\tau)}-1,$$
$$\left(3\frac{\theta_i(0|9\tau)}{\theta_i(0|\tau)}-1\right)^3=9\frac{\theta_i^4(0|3\tau)}{\theta_i^4(0|\tau)}-1.$$

for $i=2,3,4$. (see a different proof of reference to [4,p.143].

Finally we end this paper by extending this (17) and applying this generalization to derive a few well-known identities.

The identity (17) together with the following identity [17,p.324]

$$\theta_1(y)\theta_4(x)\theta_1(x+y+z)\theta_4(z)+\theta_1(x)\theta_1(z)\theta_4(y)\theta_4(x+y+z)=\theta_4\theta_4(x+z)\theta_1(y+z)\theta_1(x+y)$$

can yields an identity which is an extension of (17),

$$\frac{\theta_4'}{\theta_4}(x)+\frac{\theta_4'}{\theta_4}(y)+\frac{\theta_4'}{\theta_4}(z)-\frac{\theta_4'}{\theta_4}(x+y+z)=(\theta_1')\frac{\theta_1(x+y)\theta_1(y+z)\theta_1(x+y)}{\theta_4(x)\theta_4(y)\theta_4(z)\theta_4(x+y+z)}.$$

We can see that this identity can re-derive many identities of Ramanujan. For example, if replacing τ by $\frac{11\tau}{2}$ and setting $x=\frac{\pi\tau}{4}, y=\frac{3\pi\tau}{4}, z=\frac{5\pi\tau}{4}$, then we get

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(&\frac{q^{11n+1}}{1-q^{11n+1}}-\frac{q^{11n+3}}{1-q^{11n+3}}-\frac{q^{11n+4}}{1-q^{11n+4}}-\frac{q^{11n+5}}{1-q^{11n+5}}\\&+\frac{q^{11n+6}}{1-q^{11n+6}}+\frac{q^{11n+7}}{1-q^{11n+7}}+\frac{q^{11n+8}}{1-q^{11n+8}}-\frac{q^{11n+10}}{1-q^{11n+10}}\Big)\\&=q\prod_{n=0}^{\infty}\frac{(1-q^{11n-2})(1-q^{11n-9})(1-q^{11n})^2}{(1-q^{11n-1})(1-q^{11n-5})(1-q^{11n-6})(1-q^{11n-10})}.\end{aligned}$$

Again, we applied the imaginary transformation to (17), then we have

$$\frac{\theta_2'}{\theta_2}(x)+\frac{\theta_2'}{\theta_2}(y)-\frac{\theta_2'}{\theta_2}(x+y)=\theta_3\theta_4\frac{\theta_1(x)\theta_1(y)\theta_1(x+y)}{\theta_2(x)\theta_2(y)\theta_2(x+y)}. \tag{43}$$

In (17) and (43), replacing τ by 7τ and setting $x=\pi\tau, y=2\pi\tau$, we can obtain, respectively,

$$\frac{1}{4}\theta_2(0|\frac{\tau}{2})\theta_2(0|\frac{7\tau}{2})=\sum_{n=1}^{\infty}\frac{q^n-q^{3n}-q^{5n}+q^{9n}+q^{11n}-q^{13n}}{1-q^{14n}},$$

and

$$\theta_4(0|\tau)\theta_4(0|7\tau)=1-2\sum_{n=1}^{\infty}(-1)^n\left(\frac{q^n+q^{2n}-q^{3n}+q^{4n}-q^{5n}-q^{6n}}{1-q^{7n}}\right).$$

Both identities are known to Ramanujan in [3,p.302].

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λ_J-closed sets in generalized topological spaces

I. Arockiarani † and D. Sasikala ‡

Nirmala college for women, Coimbatore, Tamil Nadu, India
E-mail: d.sasikala@rocketmail.com

Abstract The aim of this paper is to introduce the notions of λ_J-Closed sets in generalized topological spaces and supplement their properties. We also investigate their characterizations.

Keywords J-closed set, J-open set, $gJ\lambda$-closed set, $J\lambda g$-closed set, λ_J-closed set.
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§1. Introduction and preliminaries

A. Csaszar [1] has introduced and studied generalized open sets of a set X defined in terms of monotonic functions $\gamma : P(X) \to P(X)$. $\mu = \{A \subseteq X/A \subset \gamma(A)\}$ is called the family of γ-open sets which is closed under arbitrary union and $\phi \in \mu$, then μ is called a generalized topology. Let X be a non-empty set and μ be a collection of subsets of X. Then μ is called a generalized topology (briefly GT) on X iff $\phi \in \mu$ and $G_i \in \mu$ for $i \in I \neq \phi$ implies $G = U_{i\in I} G_i \in \mu$. Generalized topological spaces are important generalizations of topological spaces and many results have been obtained by many topologist [3,4,5,11,12]. A generalized topology is said to be strong [2] if $X \in \mu$.

A space $(X,\ \mu)$ is said to be quasi-topological space [7], if μ is closed under finite intersection. The generalized closure of a subset S of X, denoted by $c_\mu(S)$, is the intersection of generalized closed sets including S and the interior of S, denoted by $i_\mu(S)$, is the union of generalized open sets contained in S.

If μ is a GT on X, $A \subseteq X$, $x \in X$, then $x \in c_\mu(A)$ iff x in $M \in \mu \Rightarrow M \cap A \neq \phi$ and $c_\mu(X/A) = X/i_\mu(A)$.

In this paper, We define a new class of sets called λ_J-closed sets in generalized topological spaces and some of their properties are established.

Definition 1.1. [6] Let(X, μ) be a GTS and $A \subseteq X$, then A is said to be

(1) μ-semi open if $A\subseteq c_\mu(i_\mu(A))$,

(2) μ-pre open if $A\subseteq i_\mu(c_\mu(A))$,

(3) μ-α open if $A\subseteq i_\mu(c_\mu i_\mu(A))$,

(4) μ-β open if $A\subseteq c_\mu(i_\mu c_\mu(A))$.

Let us denote $\sigma(\mu_X)$ (briefly σ_X or σ) the class of all μ-semi open sets on X, by $\pi(\mu_X$ or $\pi)$ the class of all μ-pre open sets, by $\alpha(\mu_X$ or $\alpha)$ the class of all μ-α open sets, by $\beta(\mu_X$ or $\beta)$ the class of all μ-β open sets.

Definition 1.2. [9] Let (X, μ) be a GTS and $A \subseteq X$, then A is said to be μ-J-open if $A\subseteq i_\mu(c_\pi(A))$.

The closure of a subset S of X, denoted by $c_J(A)$ is the intersection of generalized J-closed sets including S and the interior of S, denoted by $i_J(S)$, is the union of generalized J-open sets contained in S. The set of all J-open sets is denoted by $JO(X)$. The set all J-closed sets is denoted by $JC(X)$.

Theorem 1.1. Let $(X,\ \mu)$ be a generalized topological space. Then

(1) $c_\mu(A) = X - i_\mu(X - A)$,

(2) $i_\mu(A) = X - c_\mu(X - A)$.

Definition 1.3. [9] Let(X, λ) be a GTS and $A \subseteq X$, then A is said to be λ_α-J-closed if $c_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and $U \in JO(X)$.

Lemma 1.1. [7] Let (X, μ) be a quasi-topological space. Then $c_\mu(A \cup B) = c_\mu(A) \cup c_\mu(B)$ for every subsets A and B of X.

§2. J-Closed sets in generalized topological spaces

Definition 2.1. A subset A of a space (X, λ) is said to be $gJ\lambda$-closed if $c_\lambda(A) \subseteq U$ whenever $A \subseteq U$ and $U \in JO(X)$. The complement of $gJ\lambda$-closed set is called an $gJ\lambda$-open set.

Definition 2.2. A subset A of a space (X, λ) is said to be $J\lambda g$-closed if $c_J(A) \subseteq U$ whenever $A \subseteq U$ and $U \in JO(X)$. The complement of $J\lambda g$-closed set is called an $J\lambda g$-open set.

Theorem 2.1. Let (X, λ) be a GTS and $A \subseteq X$, then

(i) Every closed set is $gJ\lambda$-closed,

(ii) Every closed set is $J\lambda g$-closed,

(iii) Every J-closed set is $J\lambda g$-closed,

(iv) Every $gJ\lambda$-closed set is a λ_α-J-closed set,

(v) Every $J\lambda g$-closed set is a λ_α-J-closed set.

Theorem 2.2. Let (X, λ) be a quasi topological space and $A \subseteq X$. If A and B are $gJ\lambda$-closed subsets of X, then $A \cup B$ is also a $gJ\lambda$-closed set.

Proof. Suppose A and B are $gJ\lambda$-closed. Let U be a J-open set such that $A \cup B \subseteq U$. Since A and B are $gJ\lambda$-closed set, $c_\lambda(A) \subseteq U$ and $c_\lambda(B) \subseteq U$ and so $c_\lambda(A) \cup c_\lambda(B) \subseteq U$. Therefore $c_\lambda(A \cup B) \subseteq U$.

Remark 2.1. The intersection of two $gJ\lambda$-closed sets is not a $gJ\lambda$-closed set.

Example 2.1. Let $X = \{a,\ b,\ c\}$ and $\lambda = \{\phi,\ \{b\},\ X\}$. If $A = \{a,\ b\}$ and $B = \{b,\ c\}$ are $gJ\lambda$-closed sets. but $A \cap B = \{b\}$ is not a $gJ\lambda$-closed set.

Theorem 2.3. Let (X, λ) be a GTS and $A \subseteq X$. Then A is $J\lambda g$-closed if and only if $F \subseteq c_J(A) - A$ and F is J-closed implies that $F = \phi$.

Proof. Let F be a subset of J-closed subset of $c_J(A) - A$. Since $A \subseteq X$-F and A is $J\lambda g$-closed set, $c_J(A)$-$A \subseteq X$-F and so $F \subseteq X$-$c_J(A)$-A. Therefore $F = \phi$.

Conversely, U is a J-open set such that $A \subseteq U$. If $c_J(A)$ not a subset in U, then $c_J(A) \cap (X - U)$ is a non-empty J-closed subset of $c_J(A) - A$, which is a contradiction. Therefore, $c_J(A) \subseteq U$, which implies that A is $J\lambda g$-closed set.

Theorem 2.4. Let (X, λ) be a GTS. Let A and B be subsets of X. If $A \subset B \subset c_J(A)$ and A is $J\lambda g$-closed, then B is $J\lambda g$-closed.

Proof. If F is J-closed such that $F \subseteq c_J(B) - B$, therefore by hypothesis, $F \subseteq c_J(A) - A$. Since A is $J\lambda g$-closed, by theorem 2.3, $F = \phi$ and so B is $J\lambda g$-closed.

Theorem 2.5. Let (X, λ) be a GTS. Let $A \subseteq X$ be a $gJ\lambda$-closed set of X, then $c_J(A)/A$ does not contain any non-empty J-closed set.

Proof. Let F be a J-closed set of X such that $F \subseteq c_J(A)/A$. Then $F \subseteq X/A$ and hence $A \subseteq X/F \in JO(X)$. Since A is $gJ\lambda$-closed $c_J(A) \subseteq X/F$ and hence $F \subseteq X/c_J(A)$. So $F \subseteq X/c_J(A) \cap (X/c_J(A)) = \phi$.

Theorem 2.6. Let (X, λ) be a GTS. Let $A \subseteq X$ is $gJ\lambda$-open iff $F \subseteq i_\lambda(A)$ whenever F is a J-closed set such that $F \subseteq (A)$.

Proof. Let A be a $gJ\lambda$-open set of X and F be a J-closed set such that $F \subseteq (A)$. Then X/A is a $gJ\lambda$-closed set and $X/A \subseteq X/F \in JO(X)$. So $c_\lambda(X/A) = X/i_\lambda(A) \subseteq X/F$, thus $F \subseteq i_\lambda(A)$.

Conversely, let $F \subseteq i_\lambda(A)$, whenever F is J-closed such that $F \subseteq A$. Let $X/A \subseteq U$ where $U \in JO(X)$. Then $X/U \subseteq A$ and X/U is J-closed. By the assumption, $X/U \subseteq i_\lambda(A)$ and hence $c_\lambda(X/A) = X/i_\lambda(A) \subseteq U$. Hence X/A is $gJ\lambda$-closed and hence A is $gJ\lambda$-open.

Theorem 2.7. Let (X, λ) be a GTS. If A is a $gJ\lambda$-closed subset of X, then $c_\lambda(A)/A$ is $gJ\lambda$-closed.

Proof. Let A be a $gJ\lambda$-closed subset of (X, λ). Let F be a J-closed set such that $F \subseteq c_\lambda(A)/A$, so by theorem 2.5, $F = \phi$ and thus $F \subseteq i_\lambda(c_\lambda(A)/A)$. So by theorem 2.6, $c_\lambda(A)/A$ is $gJ\lambda$-closed.

§3. λ_J-Closed sets in generalized topological spaces

Definition 3.1. A subset A of $\mathcal{M}_J = \cup\{B/B \in J\}$ of a space (X, λ) is said to be λ_J-closed if $c_J(A) \cap \mathcal{M}_J \subseteq U$ whenever $A \subseteq U$ and $U \in JO(X)$. The complement of λ_J-closed set is called an λ_J-open set.

Theorem 3.1. Let X ba a non-empty set and λ be the generalized topology on X and $A \subseteq X$. Then the following properties hold:

(i)$(X - \mathcal{M}_J)$ is a J-closed set contained in every J-closed set,

(ii) $c_J(A \cap \mathcal{M}_J) \cap \mathcal{M}_J= c_J(A) \cap \mathcal{M}_J$,

(iii) If A is J-closed, then $c_J(A \cap \mathcal{M}_J) \cap \mathcal{M}_J = A \cap \mathcal{M}_J$,

(iv) $c_J(A) = (c_J(A) \cap \mathcal{M}_J) \cup (X - \mathcal{M}_J)$,

(v) If A is J-closed, then $A = (A \cap \mathcal{M}_J) \cup (X - \mathcal{M}_J)$.

Proof.

(i) If G is J-open, then $G \subseteq \mathcal{M}_J$.

(ii) We know that $c_J(A \cap \mathcal{M}_J) \cap \mathcal{M}_J \subseteq c_J(A) \cap \mathcal{M}_J$. Let $x \in c_J(A) \cap \mathcal{M}_J$. Then $x \in c_J(A)$ and $x \in \mathcal{M}_J$. Now $x \in c_J(A)$ implies that $G \cap A \notin \phi$ for every J-open set G containing x and so $G \cap (A \cap \mathcal{M}_J) \notin \phi$ for every J-open set G containing x. Therefore, $x \in c_J(A) \cap \mathcal{M}_J$, and so $x \in c_J(A) \cap \mathcal{M}_J \cap \mathcal{M}_J$. Hence $c_J(A) \cap \mathcal{M}_J \subseteq c_J(A \cap \mathcal{M}_J) \cap M_J$.

(iii) A is J-closed i.e. $c_J(A) = A$. From (ii), we have $c_J(A \cap \mathcal{M}_J) \cap \mathcal{M}_J = c_J(A) \cap \mathcal{M}_J = A \cap \mathcal{M}_J$.

(iv) $c_J(A) = c_J(A) \cap X = c_J(A) \cap (\mathcal{M}_J \cup (X - \mathcal{M}_J)) = c_J(A) \cap (\mathcal{M}_J) \cup c_J(A) \cap (X - \mathcal{M}_J) = c_J(A) \cap (\mathcal{M}_J) \cup (X - \mathcal{M}_J)$.

(v) Since A is J-closed, from (ii) we have, $c_J(A) = (c_J(A) \cap \mathcal{M}_J) \cup (X - \mathcal{M}_J)$ i.e. $A = (A \cap \mathcal{M}_J) \cup (X - \mathcal{M}_J)$.

Theorem 3.2. Let (X, λ) be a GTS and $A \subseteq X$, then the following properties hold:

(i) If A is J-closed subset of X, then $A \cap \mathcal{M}_J$ is λ_J-closed set,

(ii) $c_J(A) \cap \mathcal{M}_J$ is a λ_J-closed set for every subset A of X.

Proof.

(i) Let $A \cap \mathcal{M}_J \subseteq U$ and U be J-open. Since $c_J(A \cap \mathcal{M}_J) \cap \mathcal{M}_J = c_J(A) \cap \mathcal{M}_J$, we have $c_J(A \cap \mathcal{M}_J) \cap \mathcal{M}_J = A \cap \mathcal{M}_J \subseteq U$ and so $A \cap \mathcal{M}_J$ is λ_J-closed set.

(ii) it follows from (i).

Theorem 3.3. Let (X, λ) be a GTS. Then a subset A of M_J is λ_J-closed if and only if $F \subset c_J(A) - A$ and F is λ-J-closed implies that $F = X - \mathcal{M}_J$.

Proof. Let F be a J-closed subset of $c_J(A) - A$. Since $A \subset X - F$ and A is λ_J-closed set, $c_J(A) \cap \mathcal{M}_J \subseteq X - F$ and so $F \subset X - (c_J(A) \cap \mathcal{M}_J) = (X - c_J(A)) \cup (X - \mathcal{M}_J)$. Since $F \subset c_J(A)$, we have $F \subset (X - \mathcal{M}_J)$. Therefore $F = X - \mathcal{M}_J$.

Conversely, Let $A \subset U$ and $U \in J$. Suppose $(c_J(A) \cap \mathcal{M}_J) \cap (X - U)$ is a non empty subset. Then $(c_J(A) \cap \mathcal{M}_J) \cap (X - U) \subset (c_J(A) \cap (X - U) \subset c_J(A) \cap (X - A) \subset c_J(A) \cap A$. Thus $(c_J(A) \cap \mathcal{M}_J) \cap (X - U)$ is a non empty J-closed set contained in $c_J(A) \cap A$. Therefore, $(c_J(A) \cap \mathcal{M}_J) \cap (X - U) = \phi$, which is a contradiction. Therefore, $(c_J(A) \cap \mathcal{M}_J) \subset U$ which implies A is a λ_J-closed set.

Theorem 3.4. Let (X, λ) be a GTS. Then a λ_J-closed subset A of $\mathcal{M}_J$ is a J-closed set, if $c_J(A) - A$ is a J-closed set.

Proof. $c_J(A) - A = X - \mathcal{M} = \phi$. Then $c_J(A) = A \cup (X - \mathcal{M})$. Therefore A is a J-closed set.

Theorem 3.5. Let (X, λ) be a quasi topological space and $A \subseteq X$. If A and B are λ_J-closed subsets of X, then $A \cup B$ is also a λ_J-closed set.

Example 3.1. Let $X = \{a,\ b,\ c\}$ and $\lambda = \{\phi,\ \{a\},\ \{b\},\ \{a, b\},\ \{a, c\},\ \{b, c\},\ X\}$. Then λ is a generalized topology but not a quasi topology. $A = \{a\}$ and $B = \{b\}$ are λ_J-closed sets. but $A \cup B = \{a, b\}$ is not a λ_J-closed set.

Example 3.2. Consider the topological space (X, τ) where $X = \{a,\ b,\ c\}$ and $\lambda = \{\phi,\ \{b\},\ X\}$. If $A = \{a,\ b\}$ and $B = \{b,\ c\}$ are λ_J-closed sets, but $A \cap B = \{b\}$ is not a λ_J-closed set.

Theorem 3.6. Let (X, λ) be a GTS. If A is λ_J-closed subset of $\mathcal{M}_J$ and B is J-closed, then $A \cap B$ is a λ_J-closed set.

Proof. Suppose $A \cap B \subseteq U$ where U is J-open, then $A \subseteq (U \cup (X - B))$. Since A is λ_J-closed, $c_J(A) \cap \mathcal{M}_J \subseteq (U \cup (X - B))$ and so $c_J(A) \cap B \cap \mathcal{M}_J) = (c_J(A) \cup c_J(B)) \cap \mathcal{M}_J \subseteq U$, which implies that $(c_J(A \cup B)) \cap \mathcal{M} \subseteq U$ and so $A \cap B$ is a λ_J-closed set.

Definition 3.2. A subset A of M in a space (X, λ) is said to be λ_J-open if $\mathcal{M}_J - A$ is λ_J-closed.

Theorem 3.7. Let (X, λ) be a GTS. Let $A \subseteq \mathcal{M}_J$ is λ_J-open if and only if $F \cap \mathcal{M}_J \subseteq i_J(A)$ whenever F is J-closed and $F \cap \mathcal{M}_J \subseteq A$.

Proof. Let A be a λ_J-open subset of $\mathcal{M}_J$ and F be a J-closed subset of X such that $F \cap \mathcal{M}_J \subseteq A$. Then $\mathcal{M}_J - A \subseteq \mathcal{M}_J - (F \cap \mathcal{M}_J) = \mathcal{M}_J - F$. Since $\mathcal{M}_J - F$ is J-open and $\mathcal{M}_J - A$ is λ_J-closed, $c_J(\mathcal{M}_J - A) \cap \mathcal{M}_J \subseteq \mathcal{M}_J - F$ and so $F \subseteq \mathcal{M}_J - (c_J(\mathcal{M}_J - A) \cap \mathcal{M}_J) = \mathcal{M}_J \cap (\mathcal{M}_J - c_J(\mathcal{M}_J - A)) = i_J(A) \cap \mathcal{M}_J = i_J(A)$, which implies $F \cap \mathcal{M}_J \subseteq A$.

Conversely, Let A be a subset of $\mathcal{M}_J$ and F be a J-closed set such that $F \cap \mathcal{M}_J \subseteq A$. By hypothesis, $F \cap \mathcal{M}_J \subseteq i_J(A)$ which implies that $\mathcal{M}_J - i_J(A) \subseteq \mathcal{M}_J - (F \cap \mathcal{M}_J)$ and so $c_J(\mathcal{M}_J - A) \subseteq \mathcal{M}_J - F$. Then $c_J(\mathcal{M}_J - A) \cap \mathcal{M}_J \subseteq (\mathcal{M}_J - F) \cap \mathcal{M}_J = \mathcal{M}_J - F$ which implies that $\mathcal{M}_J - A$ is λ_J-closed and so A is λ_J-open.

Theorem 3.8. Let (X, λ) be a GTS. Let $A \subseteq \mathcal{M}_J$ is λ_J-open if and only if $U = \mathcal{M}_J$ whenever U is J-open and $i_J(A) \cup (\mathcal{M}_J - A) \subseteq U$.

Proof. Suppose A is λ_J-open and M is J-open such that $i_J(A) \cup (\mathcal{M}_J - A) \subseteq U$. Then $\mathcal{M}_J - U \subseteq (\mathcal{M}_J - i_J(A) \cap A) = c_J(\mathcal{M}_J - A) \cap A = c_J(\mathcal{M}_J - A) - (\mathcal{M}_J - A)$ and so $(\mathcal{M}_J - U) \cup (X - \mathcal{M}_J) \subseteq c_J(\mathcal{M}_J - A) \cap A$, by theorem 3.3, $(\mathcal{M}_J - U) \cup (X - \mathcal{M}_J)$ and so $(\mathcal{M}_J - U) = \phi$ which implies $\mathcal{M}_J = U$.

Conversely, Let F be a J-closed set such that $F \cap \mathcal{M}_J \subseteq A$. Since $i_J(A) \cup (\mathcal{M}_J - A) \subseteq i_J(A) \cup (\mathcal{M}_J - F) \cup (\mathcal{M}_J - \mathcal{M}_J) = i_J(A) \cup (\mathcal{M}_J - F)$ and $i_J(A) \cup (\mathcal{M}_J - F)$ is J-open, by hypothesis, $\mathcal{M}_J = i_J(A) \cup (\mathcal{M}_J - F)$ and so $F \cap \mathcal{M}_J \subseteq (i_J(A) \cup (\mathcal{M}_J - F) \cap F) = (i_J(A) \cap F) \cup (\mathcal{M}_J - F \cap F) = i_J(A) \cap F \subseteq i_J(A)$. By definition 3.2, A is λ_J-open.

Theorem 3.9. Let (X, λ) be a GTS. Let A and B be subsets of $\mathcal{M}_J$. If $i_J(A) \subseteq B \subseteq A$ and A is λ_J-open, then B is λ_J-open.

Theorem 3.10. Let (X, λ) be a GTS. Let $A \subseteq \mathcal{M}_J$ is λ_J-closed if and only $(c_J(A) - A) \cap \mathcal{M}_J$ is λ_J-open.

Proof. Suppose $(c_J(A) - A) \cap \mathcal{M}_J$ is λ_J-open. Let $A \subseteq U$ and U is J-open, since $(c_J(A) \cap (\mathcal{M}_J - U) \subseteq (c_J(A) \cap (\mathcal{M}_J - A) = (c_J(A) - A) \cap \mathcal{M}_J$. $(c_J(A) - A) \cap \mathcal{M}_J$ is λ_J-open and $(c_J(A) \cap (\mathcal{M}_J - A)$ is J-closed, by theorem 3.7, $(c_J(A) \cap (\mathcal{M}_J - U) \subseteq i_J((c_J(A) - A) \cap \mathcal{M}_J) \subseteq i_J(c_J(A) \cap i_J(\mathcal{M}_J - A)) \subseteq i_J(c_J(A) \cap i_J(X - A)) = i_J(c_J(A)) \cap (X - c_J(A)) = \phi$. Therefore $c_J(A) \cap \mathcal{M}_J \subseteq U$ which implies that A is λ_J-closed.

Conversely, suppose A is λ_J-closed and $F \cap \mathcal{M}_J \subseteq (c_J(A) - A) \cap \mathcal{M}_J$ where F is J-closed. Then $F \subseteq (c_J(A) - A)$ and by theorem 3.3, $F = X - \mathcal{M}_J$ and so $\phi = (X - \mathcal{M}_J) \cap \mathcal{M}_J = F \cap \mathcal{M}_J \subseteq (c_J(A) - A) \cap \mathcal{M}_J$ which implies that $F \cap \mathcal{M}_J \subseteq i_J(c_J(A) - A) \cap \mathcal{M}_J)$. Therefore, $c_J(A) - A$ is λ_J-open.

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Coefficient inequality for certain subclass of analytic functions

D. Vamshee Krishna † and T. Ramreddy ‡

† Department of Mathematics, GITAM Institute of Technology, GITAM University ,
Visakhapatnam-530 045, Andhra Pradesh, India
‡ Department of Mathematics, Kakatiya University, Warangal- 506 009,
Andhra Pradesh, India
E-mail: vamsheekrishna1972@gmail.com, reddytr2@yahoo.com

Abstract The objective of this paper is to introduce certain subclass of analytic functions and obtain an upper bound to the second Hankel functional $|a_2a_4 - a_3^2|$ for the functions belonging to this class, using Toeplitz determinants. The result presented here include two known results as their special cases.

Keywords Analytic function, starlike and convex functions with respect to symmetric points, convolution, upper bound, second Hankel functional, positive real function, Toeplitz determinants.
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§1. Introduction and preliminaries

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For any two analytic functions g and h respectively with their expansions as $g(z) = \sum_{k=0}^{\infty} a_k z^k$ and $h(z) = \sum_{k=0}^{\infty} b_k z^k$, the Hadamard product or convolution of $g(z)$ and $h(z)$ is defined as the power series

$$(g * h)(z) = \sum_{k=0}^{\infty} a_k b_k z^k. \tag{2}$$

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [19,20] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{3}$$

This determinant has been considered by many authors in the literature [15]. For example, Noor [16] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [11]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in$ S. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as second Hankel determinant

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{4}$$

Janteng, Halim and Darus [10] have considered the functional $|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [12]. In their work, they have shown that if $f \in$ RT then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. The same authors [9] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV, shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Mishra and Gochhayat [13] obtained the sharp bound to the non- linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2})$, by making use of the fractional differential operator due to Owa and Srivastava [17]. They have shown that, if $f \in R_\lambda(\alpha, \rho)$ then $|a_2a_4 - a_3^2| \leq \left\{\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2 cos^2\alpha}{9}\right\}$. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors $([1], [3], [14])$.

Motivated by the above mentioned results obtained by different authors in this direction, in this paper, we consider certain subclass of analytic functions and obtain an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f belonging to this class, defined as follows.

§2. Definitions and lemmas

Definition 2.1. A function $f(z) \in A$ is said to be starlike function with respect to symmetric points, if it satisfies the condition

$$Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad \forall z \in E. \tag{5}$$

The set of all such functions is denoted by ST_s, introduced and studied by Sakaguchi [25]. Further, he has shown that the functions in ST_s are close-to-convex and hence are univalent. The concept of starlike functions with respect to symmetric points have been extended to starlike functions with respect to N-symmetric points by Ratanchand [24] and Prithvipalsingh [21]. Ramreddy [22] studied the class of close-to-convex functions with respect to N-symmetric points

and proved that this class is closed under convolution with convex univalent functions.

Definition 2.2. A function $f(z) \in A$ is said to be convex function with respect to symmetric points, if it satisfies the condition,

$$Re\left\{\frac{2\{zf'(z)\}'}{\{zf'(z)+zf'(-z)\}}\right\} > 0, \quad \forall z \in E. \tag{6}$$

Definition 2.3. A function $f(z) \in A$ is said to be in the class $STCV_s(\beta)(0 \leq \beta \leq 1)$, if it satisfy the condition

$$Re\left[\frac{2\left\{zf'(z)+\beta z^2 f''(z)\right\}}{(1-\beta)\left\{f(z)-f(-z)\right\}+\beta\left\{zf'(z)+zf'(-z)\right\}}\right] > 0, \forall z \in E. \tag{7}$$

It is observed that for $\beta = 0$ and $\beta = 1$, we obtain $STCV_s(0) = ST_s$ and $STCV_s(1) = CV_s$ respectively.

We first state some preliminary lemmas required for proving our result. Let P denote the class of functions p analytic in E for which $\Re\{p(z)\} > 0$,

$$p(z) = (1 + c_1 z + c_2 z^2 + c_3 z^3 + ...) = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \forall z \in E. \tag{8}$$

Lemma 2.1. [18, 26] If $p \in P$, then $|c_k| \leq 2$, for each $k \geq 1$.

Lemma 2.2. [7] The power series for p given in (8) converges in the unit disc E to a function in P if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3, \cdots$$

and $c_{-k} = \bar{c}_k$, are all non-negative. These are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$. This necessary and sufficient condition is due to Caratheodory and and Toeplitz, can be found in [7]. We may assume without restriction that $c_1 > 0$. On using lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4c_1^2] \geq 0,$$

is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \quad \text{for some } \ x, |x| \leq 1. \tag{9}$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{10}$$

From the relations (9) and (10), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\},$$

for some real value of z with $|z| \leq 1$.

Theorem 2.1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in STCV_s(\beta)(0 \leq \beta \leq 1)$ then

$$|a_2a_4 - a_3^2| \leq \left[\frac{1}{(1+2\beta)^2}\right].$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in STCV_s(\beta)$, from the definition 1.3, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\Re\{p(z)\} > 0$ such that

$$\left[\frac{2\left\{zf'(z) + \beta z^2 f''(z)\right\}}{(1-\beta)\left\{f(z) - f(-z)\right\} + \beta\left\{zf'(z) + zf'(-z)\right\}}\right] = p(z)$$

$$\Rightarrow 2\left\{zf'(z) + \beta z^2 f''(z)\right\} = \left[(1-\beta)\left\{f(z) - f(-z)\right\} + \beta\left\{zf'(z) + zf'(-z)\right\}\right]p(z). \tag{11}$$

Replacing $f(z)$, $f'(z)$, $f''(z)$ and $p(z)$ with their equivalent series expressions in (12), we have

$$2\left[z\left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\} + \beta z^2\left\{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right\}\right]$$
$$= \left[(1-\beta)\left\{\left(z + \sum_{n=2}^{\infty} a_n z^n\right) - \left(-z + \sum_{n=2}^{\infty} a_n (-z)^n\right)\right\} + \right.$$
$$\left.\beta\left\{z\left(1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right) + z\left(1 + \sum_{n=2}^{\infty} na_n (-z)^{n-1}\right)\right\}\right] \times \left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

Upon simplification, we obtain

$$[(1 + 2(1+\beta)a_2z + 3(1+2\beta)a_3z^2 + 4(1+3\beta)a_4z^3 + \cdots)]$$
$$= [(1 + c_1z + \{c_2 + (1+2\beta)a_3\}z^2 + \{c_3 + (1+2\beta)c_1a_3\}z^3 + \cdots]. \tag{12}$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively in (13), we have

$$\{2(1+\beta)a_2 = c_1; 3(1+2\beta)a_3 = \{c_2 + (1+2\beta)a_3\}; 4(1+3\beta)a_4 = \{c_3 + (1+2\beta)c_1a_3.\}$$

After simplifying, we get

$$\{a_2 = \frac{c_1}{2(1+\beta)}; a_3 = \frac{c_2}{2(1+2\beta)}; a_4 = \frac{1}{8(1+3\beta)}(2c_3 + c_1c_2).\} \tag{13}$$

Considering the second Hankel functional $|a_2a_4 - a_3^2|$ for the function $f \in STCV_s(\beta)$ and substituting the values of a_2, a_3 and a_4 from the relation (14), we have

$$|a_2a_4 - a_3^2| = \left|\frac{c_1}{2(1+\beta)} \times \frac{1}{8(1+3\beta)}(2c_3 + c_1c_2) - \frac{c_2^2}{4(1+2\beta)^2}\right|. \tag{14}$$

Upon simplification, we obtain

$$|a_2a_4 - a_3^2| = \frac{1}{16(1+\beta)(1+2\beta)^2(1+3\beta)} \times |2(1+2\beta)^2c_1c_3 + (1+2\beta)^2c_1^2c_2 - 4(1+\beta)(1+3\beta)c_2^2|.$$

The above expression is equivalent to

$$|a_2a_4 - a_3^2| = \frac{1}{16(1+\beta)(1+2\beta)^2(1+3\beta)} \times \left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2\right|. \tag{15}$$

Where

$$\{d_1 = 2(1+2\beta)^2; d_2 = (1+2\beta)^2; d_3 = -4(1+\beta)(1+3\beta).\} \tag{16}$$

Substituting the values of c_2 and c_3 from (9) and (11) respectively from lemma 1.2 in the right hand side of (16), we have

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2| = |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + d_2c_1^2 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + d_3 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2|.$$

Using the facts $|z| < 1$ and $|xa + yb| \leq |x||a| + |y||b|$, where $x,\ y,\ a$ and b are real numbers, after simplifying, we get

$$4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2| \leq |(d_1 + 2d_2 + d_3)c_1^4 + 2d_1c_1(4-c_1^2) + 2(d_1 + d_2 + d_3)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}(4-c_1^2)|x|^2|. \tag{17}$$

Using the values of $d_1,\ d_2,\ d_3$and d_4 from the relation (17), upon simplification, we obtain

$$\{(d_1 + 2d_2 + d_3) = 4\beta^2;\ d_1 = 2(1+2\beta)^2;\ (d_1 + d_2 + d_3) = -(1+4\beta).\} \tag{18}$$

$$\left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} = 2\left\{-(2\beta^2+4\beta+1)c_1^2 + 2(1+2\beta)^2c_1 + 8(1+\beta)(1+3\beta).\right\} \tag{19}$$

Consider

$$\left\{-(2\beta^2+4\beta+1)c_1^2 + 2(1+2\beta)^2c_1 + 8(1+\beta)(1+3\beta)\right\} = (2\beta^2+4\beta+1) \times \left[-c_1^2 + \frac{2(1+2\beta)^2}{(2\beta^2+4\beta+1)}c_1 + \frac{8(1+\beta)(1+3\beta)}{(2\beta^2+4\beta+1)}\right]$$

$$= -(2\beta^2+4\beta+1) \times \left[\left\{c_1 - \frac{(1+2\beta)^2}{(2\beta^2+4\beta+1)}\right\}^2 - \frac{(1+2\beta)^4}{(2\beta^2+4\beta+1)^2} - \frac{8(1+\beta)(1+3\beta)}{(2\beta^2+4\beta+1)}\right].$$

Upon simplification, the above expression reduces to

$$= -(2\beta^2+4\beta+1) \times \left[\left\{c_1 - \frac{(1+2\beta)^2}{(2\beta^2+4\beta+1)}\right\}^2 - \left\{\frac{\sqrt{64\beta^4 + 192\beta^3 + 192\beta^2 + 72\beta} + 9}{(2\beta^2+4\beta+1)}\right\}^2\right].$$

$$\Rightarrow \left\{-(2\beta^2+4\beta+1)c_1^2+2(1+2\beta)^2c_1+8(1+\beta)(1+3\beta)\right\}$$
$$= (2\beta^2+4\beta+1)\times$$
$$\left[-c_1+\left\{\frac{(1+2\beta)^2}{(2\beta^2+4\beta+1)}+\frac{\sqrt{64\beta^4+192\beta^3+192\beta^2+72\beta}+9}{(2\beta^2+4\beta+1)}\right\}\right]$$
$$\times\left[c_1+\left\{\frac{4(1+2\beta)^2}{(2\beta^2+4\beta+1)}-\frac{\sqrt{64\beta^4+192\beta^3+192\beta^2+72\beta}+9}{(2\beta^2+4\beta+1)}\right\}\right]. \quad (20)$$

Since $c_1 \in [0,2]$, using the result $(-c_1+a)(c_1+b) \geq (-c_1-a)(-c_1-b)$, provided $a \geq b$, where $a, b \geq 0$ in the right hand side of the relation (21), upon simplification, we obtain

$$\left\{-(2\beta^2+4\beta+1)c_1^2+2(1+2\beta)^2c_1+8(1+\beta)(1+3\beta)\right\}$$
$$\geq \left\{-(2\beta^2+4\beta+1)c_1^2-2(1+2\beta)^2c_1+8(1+\beta)(1+3\beta)\right\}. \quad (21)$$

From the relations (20) and (22), we have

$$-\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}$$
$$\leq -2\left\{-(2\beta^2+4\beta+1)c_1^2-2(1+2\beta)^2c_1+8(1+\beta)(1+3\beta)\right\}. \quad (22)$$

Substituting the calculated values from (19) and (23) in the right hand side of (18), we obtain

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2| \leq |4\beta^2c_1^4+4(1+2\beta)^2c_1(4-c_1^2)-2(1+4\beta)c_1^2(4-c_1^2)|x|$$
$$-2\left\{-(2\beta^2+4\beta+1)c_1^2-2(1+2\beta)^2c_1+8(1+\beta)(1+3\beta)\right\}(4-c_1^2)|x|^2|.$$

Choosing $c_1{=}c \in [0,2]$, applying triangle inequality and replacing $|\ x\ |$ by μ in the right hand side of the above inequality, we get

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2| \leq [4\beta^2c^4+4(1+2\beta)^2c(4-c^2)+2(1+4\beta)c^2(4-c^2)\mu$$
$$+2\left\{-(2\beta^2+4\beta+1)c^2-2(1+2\beta)^2c+8(1+\beta)(1+3\beta)\right\}(4-c^2)\mu^2]$$
$$= F(c,\mu), \quad \text{for} \quad 0 \leq \mu = |x| \leq 1. \quad (23)$$

Where

$$F(c,\mu) = [4\beta^2c^4+4(1+2\beta)^2c(4-c^2)+2(1+4\beta)c^2(4-c^2)\mu$$
$$+2\left\{-(2\beta^2+4\beta+1)c^2-2(1+2\beta)^2c+8(1+\beta)(1+3\beta)\right\}(4-c^2)\mu^2]. \quad (24)$$

We next maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (25) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = [2(1+4\beta)c^2$$
$$+4\left\{-(2\beta^2+4\beta+1)c^2-2(1+2\beta)^2c+8(1+\beta)(1+3\beta)\right\}\mu]\times(4-c^2). \quad (25)$$

For $0<\mu<1$, for fixed c with $0<c<2$ and $0 \leq \beta \leq 1$, from (26), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ so that it cannot have a maximum value in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq1} F(c,\mu) = F(c,1) = G(c). \quad (26)$$

Therefore, replacing μ by 1 in (25), upon simplification, we obtain

$$G(c) = \left\{8\beta^2c^4 - 16(4\beta^2 + 4\beta + 1)c^2 + 64(1+\beta)(1+3\beta)\right\}. \tag{27}$$

$$G'(c) = \left\{32\beta^2c^3 - 32(4\beta^2 + 4\beta + 1)c\right\} = 32c\left\{(c^2-4)\beta^2 - 4\beta - 1)\right\}. \tag{28}$$

From the expression (29), we observe that $G'(c) \le 0$, for $0 \le \beta \le 1$ and $0 \le c \le 2$. Hence, $G(c)$ is a monotonically decreasing function of c in the interval $[0, 2]$, whose maximum value occurs at $c = 0$. From (28), we obtain

$$G_{max} = G(0) = 64(1+\beta)(1+3\beta). \tag{29}$$

From the relations (18) and (30), after simplifying, we get

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2| \le 16(1+\beta)(1+3\beta). \tag{30}$$

From the expressions (16) and (31), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \le \left[\frac{1}{(1+2\beta)^2}\right]. \tag{31}$$

This completes the proof of our theorem.

Proposition 2.1. Choosing $\beta = 0$, from (32), we obtain $|a_2a_4 - a_3^2| \le 1$.

Proposition 2.2. For the choice of $\beta = 1$, from (32), we get $|a_2a_4 - a_3^2| \le \frac{1}{9}$. Both the results coincide with those of Ramreddy and Vamshee Krishna [23].

Proposition 2.3. For $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in ST$, we have $|a_2a_4 - a_3^2| \le 1$. Therefore, we conclude that the upper bound to the second Hanel determinant of starlike function and a starlike function with respect to symmetric points is the same.

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Existence and uniqueness of positive solution for second order integral boundary value problem

Yonghong Ding

Department of Mathematics, Tianshui Normal University, Tianshui, PR China
E-mail: dyh198510@126.com

Abstract This paper deal with the existence and uniqueness of positive solution for the integral boundary value problem

$$\begin{cases} -u''(t) = f(t,\ u(t)), & t \in [0,\ 1]; \\ au(0) - bu'(0) = \int_0^1 g(s)u(s)ds, & cu(1) + du'(1) = \int_0^1 h(s)u(s)ds. \end{cases}$$

The discussion is based on the method of lower and upper solutions and maximal principle.

Keywords Lower and upper solution, fixed point, maximal principle, positive solutions.
2000 Mathematics Subject Classification: 34B15, 34B18

§1. Introduction

We consider the integral boundary value problem

$$\begin{cases} -u''(t) = f(t,\ u(t)), & t \in [0,\ 1]; \\ au(0) - bu'(0) = \int_0^1 g(s)u(s)ds, & cu(1) + du'(1) = \int_0^1 h(s)u(s)ds, \end{cases} \tag{1}$$

where $f : [0,\ 1] \times [0,\ +\infty) \to [0,\ +\infty)$ is continuous. $g,\ h \in L^1[0,\ 1]$ are nonnegative, $a,\ b,\ c,\ d \geqslant 0$ and $\rho = ac + ad + bc > 0$.

The (1) arises in many different areas of applied mathematics and physics, and only its positive solution is significant in some practice. For the special cases of (1), the existence of positive solutions has been widely investigate by many authors. For details and reference see [1-7]. The main tools used in these papers are fixed point theorems. However, to the author's knowledge, few papers can be found in the literature on existence of positive solutions for integral boundary value problem by using lower and upper solution method.

In [8], the authors used the method of lower and upper solutions to research second order multi-point boundary value problem. Being directly motivated by [8], in this paper, by constructing a pair of lower and upper solution and combine with maximal principle, we will show that (1) has a uniqueness positive solution under that $f(t,\ u)$ is decreasing on u.

§2. Preliminaries

For convenience, we denote

$$H=\int_0^1\frac{as+b}{\rho}h(s)ds,\quad G=\int_0^1\frac{c+d-cs}{\rho}g(s)ds,\quad C=\frac{1}{1-H},\quad D=\frac{1}{1-G},$$

$$K=\int_0^1\frac{c+d-cs}{\rho}h(s)ds,\quad F=\int_0^1\frac{as+b}{\rho}g(s)ds,\quad R=\int_0^1 h(s)ds,\quad S=\int_0^1 g(s)ds,$$

$$N=\frac{(a+b)(c+d)}{\rho},\quad M=1+\frac{[C(a+b)+(d+c)CDF]R}{\rho(1-CKDF)}+\frac{[CKD(a+b)+(d+c)D]S}{\rho(1-CKDF)},$$

$$A(t)=\frac{C(at+b)+(d+c-ct)CDF}{\rho(1-CKDF)},\quad B(t)=\frac{CKD(at+b)+(d+c-ct)D}{\rho(1-CKDF)}.$$

We assume the following condition throughout:

$$\text{(H1)}\qquad H,\ G\in[0,\ 1),\ CKDF\in[0,\ 1).$$

Definition 2.1. A function $\alpha\in C^2[0,\ 1]$ is called a lower solution of the (1) if it satisfies

$$\begin{cases}-\alpha''(t)\le f(t,\ \alpha(t)), & t\in[0,\ 1];\\ a\alpha(0)-b\alpha'(0)\le\int_0^1 g(s)\alpha(s)ds, & c\alpha(1)+d\alpha'(1)\le\int_0^1 h(s)\alpha(s)ds.\end{cases}$$

Analogously, a function $\beta\in C^2[0,\ 1]$ is called a upper solution of the (1) if it satisfies the reversed inequalities. In order to prove our main results, we need the following maximal principle.

Lemma 2.1. Assume that (H1) holds. Let $u\in C^2[0,\ 1]$ and satisfies

$$\begin{cases}-u''(t)\ge 0, & t\in[0,\ 1];\\ au(0)-bu'(0)\ge\int_0^1 g(s)u(s)ds, & c\alpha(1)+d\alpha'(1)\ge\int_0^1 h(s)u(s)ds,\end{cases}\tag{2}$$

then $u(t)\ge 0,\ t\in[0,\ 1]$.

Proof. Let

$$\begin{cases}-u''(t)=y(t), & t\in[0,\ 1];\\ au(0)-bu'(0)-\int_0^1 g(s)u(s)ds=r_1, & cu(1)+du'(1)-\int_0^1 h(s)u(s)ds=r_2,\end{cases}\tag{3}$$

then $r_1\ge 0,\ r_2\ge 0,\ y(t)\ge 0,\ t\in[0,\ 1]$.

By integrating (3) on $[0,\ t]$, we have

$$u'(t)=-\int_0^t y(s)ds+u'(0).\tag{4}$$

Thus,

$$u(t)=-\int_0^t(t-s)y(s)ds+u'(0)t+u(0).\tag{5}$$

Let $t=1$ in (4) and (5), then

$$u'(1)=-\int_0^1 y(s)ds+u'(0),\tag{6}$$

$$u(1)=-\int_0^1(1-s)y(s)ds+u'(0)+u(0). \tag{7}$$

According to the boundary condition $cu(1)+du'(1)-\int_0^1 h(s)u(s)ds=r_2$, we have

$$-c\int_0^1(1-s)y(s)ds+cu'(0)+cu(0)-d\int_0^1 y(s)ds+du'(0)-\int_0^1 h(s)u(s)ds=r_2.$$

By $u(0)=\frac{1}{a}\big(bu'(0)+\int_0^1 g(s)u(s)ds+r_1\big)$,

$$u'(0)=\frac{a}{\rho}[c\int_0^1(1-s)y(s)ds+d\int_0^1 y(s)ds-\frac{c}{a}\int_0^1 g(s)u(s)ds+\int_0^1 h(s)u(s)ds-\frac{c}{a}r_1+r_2], \tag{8}$$

therefore,

$$u(0)=\frac{b}{\rho}[c\int_0^1(1-s)y(s)ds+d\int_0^1 y(s)ds-\frac{c}{a}\int_0^1 g(s)u(s)ds+\int_0^1 h(s)u(s)ds-\frac{c}{a}r_1+r_2]$$
$$+\frac{1}{a}\int_0^1 g(s)u(s)ds+\frac{1}{a}r_1. \tag{9}$$

From (5), (8) and (9), we have

$$u(t)=\int_0^1 G(t,s)y(s)ds+\int_0^1\frac{at+b}{\rho}h(s)u(s)ds+\int_0^1\frac{d+c-ct}{\rho}g(s)u(s)ds$$
$$+\frac{cr_1(1-t)+ar_2t+br_2+dr_1}{\rho}, \tag{10}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}(b+as)(d+c-ct), & 0\le s\le t\le 1;\\ (b+at)(d+c-cs), & 0\le t\le s\le 1.\end{cases}$$

Then, from (10),

$$\int_0^1 h(s)u(s)ds=\int_0^1 h(s)\big[\int_0^1 G(s,\tau)y(\tau)d\tau+\int_0^1\frac{as+b}{\rho}h(\tau)u(\tau)d\tau+\int_0^1\frac{d+c-cs}{\rho}g(\tau)u(\tau)d\tau$$
$$+\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}\big]ds, \tag{11}$$

$$\int_0^1 g(s)u(s)ds=\int_0^1 g(s)\big[\int_0^1 G(s,\tau)y(\tau)d\tau+\int_0^1\frac{as+b}{\rho}h(\tau)u(\tau)d\tau+\int_0^1\frac{d+c-cs}{\rho}g(\tau)u(\tau)d\tau$$
$$+\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}\big]ds. \tag{12}$$

Hence, from (11) and (12),

$$\int_0^1 h(s)u(s)ds=\frac{1}{1-CKDF}\big[C\int_0^1\int_0^1 h(s)G(s,\tau)y(\tau)d\tau ds+CKD\int_0^1\int_0^1 g(s)G(s,\tau)y(\tau)d\tau ds$$
$$+C\int_0^1 h(s)\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}ds+CKD\int_0^1 g(s)\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}ds\big]. \tag{13}$$

and

$$\int_0^1 g(s)u(s)ds = \frac{1}{1-CKDF}\big[CDF\int_0^1\int_0^1 h(s)G(s,\tau)y(\tau)d\tau ds + D\int_0^1\int_0^1 g(s)G(s,\tau)y(\tau)d\tau ds$$

$$+CDF\int_0^1 h(s)\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}ds + D\int_0^1 g(s)\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}ds\big]. \tag{14}$$

From (10), (13) and (14), we get

$$u(t) = \int_0^1 G(t,s)y(s)ds + A(t)\int_0^1\int_0^1 h(s)G(s,\tau)y(\tau)d\tau ds + B(t)\int_0^1\int_0^1 g(s)G(s,\tau)y(\tau)d\tau ds$$

$$+A(t)\int_0^1 h(s)\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}ds + B(t)\int_0^1 g(s)\frac{cr_1(1-s)+ar_2s+br_2+dr_1}{\rho}ds$$

$$+\frac{cr_1(1-t)+ar_2t+br_2+dr_1}{\rho}.$$

Therefore $u(t)\geq 0,\ t\in[0,\ 1]$.

Lemma 2.2. Assume (H1) holds. Let $\varphi\in C[0,\ 1]$, then boundary value problem

$$\begin{cases} -u''(t)=\varphi(t), & t\in[0,\ 1]; \\ au(0)-bu'(0)=\int_0^1 g(s)u(s)ds, & cu(1)+du'(1)=\int_0^1 h(s)u(s)ds, \end{cases} \tag{15}$$

has a unique solution $u(t)$ which is given by

$$u(t)=\int_0^1 \gamma(t,\ s)\varphi(s)ds,$$

where

$$\gamma(t,\ s)=G(t,s)+A(t)\int_0^1 G(s,\tau)h(\tau)d\tau + B(t)\int_0^1 G(s,\tau)g(\tau)d\tau.$$

Proof. First suppose that $u\in C[0,\ 1]$ is a solution of BVP(15). Choose $r_1=r_2=0$ in lemma 2.1, we can get

$$u(t) = \int_0^1 G(t,s)\varphi(s)ds + A(t)\int_0^1\int_0^1 h(s)G(s,\tau)\varphi(\tau)d\tau ds + B(t)\int_0^1\int_0^1 g(s)G(s,\tau)\varphi(\tau)d\tau ds$$

$$=\int_0^1 \gamma(t,\ s)\varphi(s)ds.$$

Conversely, suppose $u(t)=\int_0^1\gamma(t,\ s)\varphi(s)ds$, then

$$u''(t)=\frac{1}{\rho}[-c(at+b)\varphi(t)-a(d+c-ct)\varphi(t)]=-\varphi(t).$$

By computation, we have

$$au(0)-bu'(0)=\int_0^1 g(s)u(s)ds, \quad cu(1)+du'(1)=\int_0^1 h(s)u(s)ds.$$

Thus $u(t)$ is a solution of BVP(15).

Lemma 2.3. Suppose (H1) holds. Then

(i) $\delta G(t,\ t)G(s,\ s) \le G(t,\ s) \le G(s,\ s) \le N, \quad \forall s,\ t \in [0,\ 1]$;

(ii) $\sigma z(t)G(t,\ t) \le \gamma(t,\ s) \le MG(s,\ s), \quad \forall s,\ t \in [0,\ 1]$,

where

$$\delta = \frac{\rho}{(a+b)(c+d)}, \qquad \sigma = \delta\big(1 + \int_0^1 G(\tau,\tau)h(\tau)d\tau + \int_0^1 G(\tau,\tau)g(\tau)d\tau\big),$$

$$z(t) = \min\{G(t,\ t),\ A(t),\ B(t)\}.$$

Proof. (i) From the expression of $G(t,\ s)$ we see that

$$G(t,\ s) \le G(s,\ s) \le N, \quad \forall s,\ t \in [0,\ 1].$$

Furthermore, we have

$$\frac{G(t,\ s)}{G(s,\ s)G(t,\ t)} = \frac{\rho}{(a+bs)(c+d-cs)} \ge \frac{\rho}{(a+b)(c+d)} = \delta.$$

So $\delta G(t,\ t)G(s,\ s) \le G(t,\ s)$. Therefore, (i) holds.

(ii) On the one hand,

$$\begin{aligned}
\gamma(t,\ s) &\ge \delta G(t,\ t)G(s,\ s) + A(t)\int_0^1 \delta G(s,\ s)G(\tau,\tau)h(\tau)d\tau + B(t)\int_0^1 \delta G(s,\ s)G(\tau,\tau)g(\tau)d\tau \\
&= \delta G(s,\ s)\big(G(t,\ t) + A(t)\int_0^1 G(\tau,\tau)h(\tau)d\tau + B(t)\int_0^1 G(\tau,\tau)g(\tau)d\tau\big) \\
&\ge \delta z(t)G(s,\ s)\big(1 + \int_0^1 G(\tau,\tau)h(\tau)d\tau + \int_0^1 G(\tau,\tau)g(\tau)d\tau\big) \\
&= \sigma z(t)G(s,\ s).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma(t,\ s) &\le G(s,\ s) + A(t)\int_0^1 G(s,\ s)h(\tau)d\tau + B(t)\int_0^1 G(s,\ s)g(\tau)d\tau \\
&= G(s,\ s)\big(1 + A(t)\int_0^1 h(\tau)d\tau + B(t)\int_0^1 g(\tau)d\tau\big) \\
&= MG(s,\ s).
\end{aligned}$$

§3. Main results

Theorem 3.1. Assume (H1) holds. If f satisfies the following condition:

(H2) $f : [0,\ 1] \times [0,\ +\infty) \to [0,\ +\infty)$ is continuous, $f(t,u) \not\equiv 0$, and $f(t,u)$ is decreasing on u. Then (1) has a unique positive solution.

Proof. Based on the preceding preliminaries, we can divide our proof into three steps:

Step 1. We first construct a pair of lower and upper solution of (1). Let

$$a(t) = \int_0^1 \gamma(t,\ s)f(s,\ N)ds, \qquad b(t) = \int_0^1 \gamma(t,\ s)f(s,\ z(s))ds.$$

Since $f(t,\ u)$ is decreasing on $u,\ t\in[0,\ 1]$, and by lemma 2.3, we obtain

$$\sigma z(t)\int_0^1 G(s,\ s)f(s,\ N)ds\le a(t)\le b(t)\le M\int_0^1 G(s,\ s)f(s,\ z(s))ds.$$

Denote

$$k_1=M\int_0^1 G(s,\ s)f(s,\ z(s))ds,\qquad k_2=\sigma\int_0^1 G(s,\ s)f(s,\ N)ds,$$

$$l=\min\{1,\ \frac{1}{k_1},\ \frac{1}{k_2}\},\qquad L=\max\{1,\ \frac{1}{k_1},\ \frac{1}{k_2}\}.$$

It is easy to see that

$$l\le 1,\quad lk_1\le 1,\quad L\ge 1,\quad Lk_2\ge 1,$$

and

$$la(t)\le lk_1\le 1\le N,\qquad z(t)\le Lk_2z(t)\le Lb(t).$$

Let $\alpha(t)=la(t),\ \beta(t)=Lb(t)$, then

$$\begin{aligned}-\alpha''(t)-f(t,\ \alpha(t)) &= lf(t,\ N)-f(t,\ la(t))\\ &\le lf(t,\ la(t))-f(t,\ la(t))\\ &\le 0,\end{aligned}$$

$$\begin{aligned}-\beta''(t)-f(t,\ \beta(t)) &= Lf(t,\ z(t))-f(t,\ Lb(t))\\ &\ge Lf(t,\ Lb(t))-f(t,\ Lb(t))\\ &\ge 0.\end{aligned}$$

By calculations we have

$$a\alpha(0)-b\alpha'(0)=\int_0^1 g(s)\alpha(s)ds,\ c\alpha(1)+d\alpha'(1)=\int_0^1 h(s)\alpha(s)ds,$$

$$a\beta(0)-b\beta'(0)=\int_0^1 g(s)\beta(s)ds,\ c\beta(1)+d\beta'(1)=\int_0^1 h(s)\beta(s)ds.$$

Therefore, $\alpha(t)$ is a lower solution of (1), and $\beta(t)$ is a upper solution of (1).

Step 2. We prove (1) has a positive solution $\widetilde{u}(t)$ satisfying $\alpha(t)\le\widetilde{u}(t)\le\beta(t)$.

For any $x(t)\in C[0,\ 1]$, define

$$F(t,\ x)=\begin{cases} f(t,\ \alpha(t)), & x<\alpha(t);\\ f(t,\ x(t)), & \alpha(t)\le x\le\beta(t);\\ f(t,\ \beta(t)), & x>\beta(t).\end{cases}$$

Then $F:[0,\ 1]\times[0,\ +\infty)\times\to[0,\ +\infty)$ is continuous and bounded. Consider the boundary value problem

$$\begin{cases} -u''(t)=F(t,\ u(t)), & t\in[0,\ 1];\\ au(0)-bu'(0)=\int_0^1 g(s)u(s)ds, & cu(1)+du'(1)=\int_0^1 h(s)u(s)ds.\end{cases}\tag{16}$$

We prove (16) has at least one solution.

Define the operator $T : C[0,\ 1] \to C[0,\ 1]$ as follows:

$$(Tu)(t) = \int_0^1 \gamma(t,\ s)F(s,\ u(s))ds.$$

By lemma 2.2, each fixed point of T is a solution for (16). It is well known that $T : C[0,\ 1] \to C[0,\ 1]$ is completely continuous and F is bounded, by using Schauder fixed point theorem, we obtain that T has fixed point $\widetilde{u} \in C[0,\ 1]$. In the following we show that $\alpha(t) \leq \widetilde{u}(t) \leq \beta(t)$, this means that $F(t,\ \widetilde{u}(t)) = f(t,\ \widetilde{u}(t))$, hence, $\widetilde{u}(t)$ is also a solution of (1).

Assume $\alpha(t) \nleq \widetilde{u}(t)$, then there exists $t_0 \in [0,\ 1]$ such that $\alpha(t_0) > \widetilde{u}(t_0)$. Since $\alpha(t)$ and $\widetilde{u}(t)$ are continuous, there exists a subset $[t_1,\ t_2] \subset [0,\ 1]$ with $t_0 \in (t_1,\ t_2)$ such that $\alpha(t) > \widetilde{u}(t),\ t \in [t_1,\ t_2]$.

Set

$$w(t) = \widetilde{u}(t) - \alpha(t),$$

$$a_1 = \inf\left\{t_1 |\ \exists [t_1,\ t_2] \in [0,\ 1],\ t_0 \in (t_1,\ t_2) \text{ such that } w(t) < 0 \text{ for } t \in [t_1,\ t_2]\right\},$$

$$b_1 = \sup\left\{t_2 |\ \exists [t_1,\ t_2] \in [0,\ 1],\ t_0 \in (t_1,\ t_2) \text{ such that } w(t) < 0 \text{ for } t \in [t_1,\ t_2]\right\}.$$

Then

$$w(t) < 0, \qquad t \in (a_1,\ b_1), \tag{17}$$

and

$$w(a_1) = w(b_1) = 0,\ w'(a_1) \leq 0,\ w'(b_1) \geq 0.$$

On the other hand, for any $t \in [a_1,\ b_1]$, we have

$$\begin{aligned} -w''(t) &= -\widetilde{u}''(t) + \alpha''(t) \\ &= F(t,\ \widetilde{u}(t)) + \alpha''(t) \\ &= f(t,\ \alpha(t)) + \alpha''(t) \\ &\geq 0. \end{aligned}$$

and

$$aw(a_1) - bw'(a_1) \geq \int_0^1 g(s)w(s)ds, \qquad cw(b_1) + dw'(b_1) \geq \int_0^1 h(s)w(s)ds.$$

By lemma 2.1, we have $w(t) \geq 0,\ \ t \in [a_1,\ b_1]$, which contradicts (17). Thus $\alpha(t) \leq \widetilde{u}(t)$.

Similarly, we assert that $\widetilde{u}(t) \leq \beta(t)$. From assumption (H2), it follows that $\alpha(t) > 0$. Therefore, $\widetilde{u}(t)$ is a positive solution of (1).

Step 3. We prove uniqueness. Suppose $\overline{u}(t)$ is also a positive solution of (1.1) and $\overline{u}(t) \not\equiv \widetilde{u}(t)$. Then there exists $\xi \in [0,\ 1]$ such that $\overline{u}(\xi) \neq \widetilde{u}(\xi)$, without loss of generality, we assume that $\overline{u}(\xi) < \widetilde{u}(\xi)$. Set $v(t) = \overline{u}(t) - \widetilde{u}(t)$, it is similar to step 2, we can get a contradiction. So $\widetilde{u}(t)$ is a unique positive solution.

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Smarandache bisymmetric geometric determinat sequences

Aldous Cesar F. Bueno

Department of Mathematics and Physics, Central Luzon State University,
Science City of Muñoz 3120, Nueva Ecija, Philippines
Email: aldouz_cezar@yahoo.com

Abstract In this paper, the Smarandache bisymmetric geometric determinant sequence was defined and the formula for its n-th term was obtained.
Keywords Smarandache bisymmetric geometric determinant sequence.

§1. Introduction and preliminaries

Majumdar [1] gave the formula for n-th term of the following sequences: Smarandache cyclic natural determinant sequence, Smarandache cyclic arithmetic determinant sequence, Smarandache bisymmetric natural determinant sequence and Smarandache bisymmetric arithmetic determinant sequence.

Definition 1.1. The Smarandache bisymmetric geometric determinant sequence denoted by $\{SBGDS(n)\}$ is given by

$$\{SBGDS(n)\} = \left\{ |a|, \begin{vmatrix} a & ar \\ ar & a \end{vmatrix}, \cdots, \begin{vmatrix} a & ar & ar^2 & \cdots & ar^{n-2} & ar^{n-1} \\ ar & ar^2 & ar^3 & \cdots & ar^{n-1} & ar^{n-2} \\ ar^2 & ar^3 & ar^4 & \cdots & ar^{n-2} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^{n-2} & ar^{n-1} & ar^{n-2} & \cdots & ar^2 & ar \\ ar^{n-1} & ar^{n-2} & ar^{n-3} & \cdots & ar & a \end{vmatrix}, \cdots \right\}.$$

For the rest of this paper, let $|A|$ be the notation for the determinant of a matrix A.

§2. Preliminary result

Lemma 2.1.

$$|K| = \begin{vmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ r & r^2 & r^3 & \cdots & r^{n-1} & r^{n-2} \\ r^2 & r^3 & r^4 & \cdots & r^{n-2} & r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-2} & r^{n-1} & r^{n-2} & \cdots & r^2 & r \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1 \end{vmatrix} = (-1)^{\lfloor \frac{n-1}{2} \rfloor} \left(r^n - r^{n-2}\right)^{n-1},$$

where $\lfloor x \rfloor$ is the floor function.

Proof.

$$K = \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ r & r^2 & r^3 & \cdots & r^{n-1} & r^{n-2} \\ r^2 & r^3 & r^4 & \cdots & r^{n-2} & r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-2} & r^{n-1} & r^{n-2} & \cdots & r^2 & r \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ 0 & 0 & 0 & \cdots & 0 & r^n - r^{n-2} \\ 0 & 0 & 0 & \cdots & r^n - r^{n-2} & r^{n+1} - r^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & r^n - r^{n-2} & \cdots & r^{2n-4} - r^2 & r^{2n-3} - r \\ 0 & r^n - r^{n-2} & r^{n+1} - r^{n-1} & \cdots & r^{2n-3} - r & r^{2n-2} - 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \\ 0 & r^n - r^{n-2} & r^{n+1} - r^{n-1} & \cdots & r^{2n-3} - r & r^{2n-2} - 1 \\ 0 & 0 & r^n - r^{n-2} & \cdots & r^{2n-4} - r^2 & r^{2n-3} - r \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r^n - r^{n-2} & r^{n+1} - r^{n-3} \\ 0 & 0 & 0 & \cdots & 0 & r^n - r^{n-2} \end{pmatrix}.$$

Via the operations $r^k\, R_1 - R_{k+1} \to R_{k+1}$ where $k = 1, 2, \cdots, n-1$ and followed by $R_n \leftrightarrow R_2,\ R_{n-1} \leftrightarrow R_3$, and so on and there will be a total of $\lfloor \frac{n-1}{2} \rfloor$ inversions. Solving for the determinant of the last matrix multiplied to $(-1)^{\lfloor \frac{n-1}{2} \rfloor}$ gives the desired result.

§3. Main result

Theorem 3.1.

$$SBGDS(n) = \begin{cases} a, & \text{if} \quad n = 1; \\ a^2\left(1 - r^2\right), & \text{if} \quad n = 2; \\ (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} a^n \left(r^n - r^{n-2}\right)^{n-1}, & \text{if} \quad n > 2. \end{cases}$$

Proof. The cases for $n = 1$ and $n = 2$ are trivial. For $n > 2$,

$$SBGDS(n) = |aK| = a^n\,|K| = (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} a^n \left(r^n - r^{n-2}\right)^{n-1},$$

via the lemma 2.1.

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Certain integral involving inverse hyperbolic function

Salahuddin

Mewar University, Gangrar, Chittorgarh(Rajasthan), India
E-mails: vsludn@gmail.com, sludn@yahoo.com

Abstract In this paper we have developed some indefinite integrals involving inverse Hyperbolic function. The results represent here are assumed to be new.

Keywords Hypergeometric function , elliptic integral.
2000 Mathematics Subject Classification: 33D50,33D60,33C75,33E05

§1. Introduction and preliminaries

In integral calculus, elliptic integrals originally arose in connection with the problem of giving the arc length of an ellipse. They were first studied by Giulio Fagnano and Leonhard Euler. Modern mathematics defines an "elliptic integral" as any function f which can be expressed in the form

$$f(x) = \int_c^x R\Big(t, \sqrt{P(t)}\Big)dt, \tag{1.1}$$

where R is a rational function of its two arguments, P is a polynomial of degree 3 or 4 with no repeated roots, and c is a constant.

In general, elliptic integrals cannot be expressed in terms of elementary functions. Exceptions to this general rule are when P has repeated roots, or when $R(x, y)$ contains no odd powers of y. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e. the elliptic integrals of the first, second and third kind).

Besides the Legendre form, the elliptic integrals may also be expressed in Carlson symmetric form. Additional insight into the theory of the elliptic integral may be gained through the study of the Schwarz-Christoffel mapping. Historically, elliptic functions were discovered as inverse functions of elliptic integrals.

Incomplete elliptic integrals are functions of two arguments, complete elliptic integrals are functions of a single argument.

Definition 1.1. The incomplete elliptic integral of the first kind F is defined as

$$F(\psi, k) = F(\psi \mid k^2) = F(\sin\psi; k) = \int_0^{\psi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{1.2}$$

This is the trigonometric form of the integral, substituting $t = \sin\theta$, $x = \sin\psi$, one obtains Jacobi's form:

$$F(x;k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1.3}$$

Equivalently, in terms of the amplitude and modular angle one has:

$$F(\psi\backslash\alpha) = F(\psi, \sin\alpha) = \int_0^\psi \frac{d\theta}{\sqrt{1-(\sin\theta\sin\alpha)^2}}. \tag{1.4}$$

In this notation, the use of a vertical bar as delimiter indicates that the argument following it is the "parameter" (as defined above), while the backslash indicates that it is the modular angle. The use of a semicolon implies that the argument preceding it is the sine of the amplitude:

$$F(\psi, \sin\alpha) = F(\psi \mid \sin^2\alpha) = F(\psi\backslash\alpha) = F(\sin\psi; \sin\alpha). \tag{1.5}$$

Definition 1.2. Incomplete elliptic integral of the second kind E is defined as

$$E(\psi, k) = E(\psi \mid k^2) = E(\sin\psi; k) = \sqrt{1-k^2\sin^2\theta}\ d\theta. \tag{1.6}$$

Substituting $t = \sin\theta$ and $x = \sin\psi$, one obtains Jacobi's form:

$$E(x;k) = \int_0^x \frac{\sqrt{1-k^2t2}}{\sqrt{1-t^2}}\ dt. \tag{1.7}$$

Equivalently, in terms of the amplitude and modular angle:

$$E(\psi\backslash\alpha) = E(\psi, \sin\alpha) = \int_0^\psi \sqrt{1-(\sin\theta\sin\alpha)^2}\ d\theta. \tag{1.8}$$

Definition 1.3. Incomplete elliptic integral of the third kind Π is defined as

$$\Pi(n;\psi\backslash\alpha) = \int_0^\psi \frac{1}{1-n\sin^2\theta}\ \frac{d\theta}{1-(\sin\theta\sin\alpha)^2}, \tag{1.9}$$

or

$$\Pi(n;\psi \mid m) = \int_0^{\sin\psi} \frac{1}{1-nt^2}\ \frac{dt}{(1-mt^2)(1-t^2)}. \tag{1.10}$$

The number n is called the characteristic and can take on any value, independently of the other arguments.

Definition 1.4. Elliptic Integrals are said to be 'complete' when the amplitude $\psi = \frac{\pi}{2}$ and therefore $x = 1$.

The complete elliptic integral of the first kind K may thus be defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{1.11}$$

or more compactly in terms of the incomplete integral of the first kind as

$$K(k) = F\Big(\frac{\pi}{2}, k\Big) = F(1;k). \tag{1.12}$$

It can be expressed as a power series

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[P_{2n}(0)\right]^2 k^{2n}, \tag{1.13}$$

where P_n is the Legendre polynomial, which is equivalent to

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\ .\ 3}{2\ .\ 4}\right)^2 k^4 + \cdots + \left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2 k^{2n} + \cdots\right], \tag{1.14}$$

where $n!!$ denotes the double factorial.

In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$K(k) = \frac{\pi}{2}\ {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; k^2\Big). \tag{1.15}$$

The complete elliptic integral of the first kind is sometimes called the quarter period. It can most efficiently be computed in terms of the arithmetic-geometric mean:

$$K(k) = \frac{\frac{\pi}{2}}{agm(1-k, 1+k)}. \tag{1.16}$$

The complete elliptic integral of the second kind E is proportional to the circumference of the ellipse C:

$$C = 4aE(e),$$

where a is the semi-major axis, e is the eccentricity, and E may be defined as

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\ d\theta = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\ dt, \tag{1.17}$$

or more compactly in terms of the incomplete integral of the second kind as

$$E(k) = E\Big(\frac{\pi}{2}, k\Big) = E(1; k). \tag{1.18}$$

It can be expressed as a power series

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{1-2n}, \tag{1.19}$$

which is equivalent to

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\ .\ 3}{2\ .\ 4}\right)^2 \frac{k^4}{3} - \cdots - \left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2 \frac{k^{2n}}{2n-1} - \cdots\right]. \tag{1.20}$$

In terms of the Gauss hypergeometric function, the complete elliptic integral of the second kind can be expressed as

$$E(k) = \frac{\pi}{2}\ {}_2F_1\Big(\frac{1}{2}, -\frac{1}{2}; 1; k^2\Big). \tag{1.21}$$

The complete elliptic integral of the third kind Π can be defined as

$$\Pi(n,k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}. \tag{1.22}$$

Definition 1.5. A generalized hypergeometric function ${}_pF_q(a_1, \cdots a_p; b_1, \cdots b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(K+b_2)\cdots(k+b_q)(k+1)}\ z, \tag{1.23}$$

where $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{lr} a_1, a_2, \cdots, a_p \quad ; & \\ & z \\ b_1, b_2, \cdots, b_q \quad ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k z^k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!}, \tag{1.24}$$

or

$${}_pF_q\left[\begin{array}{lr} (a_p) \quad ; & \\ & z \\ (b_q) \quad ; & \end{array}\right] \equiv {}_pF_q\left[\begin{array}{lr} (a_j)_{j=1}^p \quad ; & \\ & z \\ (b_j)_{j=1}^q \quad ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_p))_k z^k}{((b_q))_k k!}, \tag{1.25}$$

where the parameters $b_1, b_2, \cdots, b_q$ are neither zero nor negative integers and p, q are non-negative integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $|\ z\ | < 1$ if $p \neq q+1$, diverges for all z, $z \neq 0$ if $p > q+1$.

The ${}_pF_q$ series absolutely converges for $|\ z\ | = 1$ if $R(\zeta) < 0$, conditionally converges for $|\ z\ | = 1, z \neq 0$ if $0 \leq R(\zeta) < 1$, diverges for $|\ z\ | = 1$, if $1 \leq R(\zeta)$, $\zeta = \sum\limits_{i=1}^{p} a_i - \sum\limits_{i=0}^{q} b_i$.

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2$, $q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0. \tag{1.26}$$

The solution of this equation is

$$y = A_0\Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\Big]. \tag{1.27}$$

This is the so-called regular solution, denoted

$${}_2F_1(a, b; c; z) = \Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\Big] = \sum_{k=0}^{\infty} \frac{(a)_k\ (b)_k z^k}{(c)_k k!}, \tag{1.28}$$

which converges if c is not a negative integer for all of $|\ z\ | < 1$ and on the unit circle $|\ z\ | = 1$ if $R(c-a-b) > 0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

§2. Main integrals

$$\int \frac{d\phi}{\sqrt{1+x\sinh^{-1}\phi}} = \frac{1}{2}\Big[(-\sinh^{-1}\phi)^{-m}\sinh^{-1}\phi^{m}\ \Gamma(m+1,-\sinh^{-1}\phi)$$

$$-\Gamma(m+1,\sinh^{-1}\phi)\Big]\ {}_1F_0\Big(\frac{1}{2};-;-x\Big) + \text{Constant}. \tag{2.1}$$

$$\int \frac{d\phi}{\sqrt{1+x\cosh^{-1}\phi}} = \frac{1}{2}\Big[(-\cosh^{-1}\phi)^{-m}\cosh^{-1}\phi^{m}\ \Gamma(m+1,-\cosh^{-1}\phi)+$$

$$\Gamma(m+1,\cosh^{-1}\phi)\Big]\ {}_1F_0\Big(\frac{1}{2};-;-x\Big) + \text{Constant}. \tag{2.2}$$

$$\int \frac{d\phi}{\sqrt{1+x\sin^{-1}\phi}} = \frac{1}{2}\iota\sin^{-1}\phi^{m}\{\sin^{-1}\phi^{2}\}^{-m}\Big[(-\iota\sin^{-1}\phi)^{m}\ \Gamma(m+1,\iota\sin^{-1}\phi)-$$

$$-(\iota\sin^{-1}\phi)^{m}\Gamma(m+1,-\iota\sin^{-1}\phi)\Big]\ {}_1F_0\Big(\frac{1}{2};-;-x\Big) + \text{Constant}. \tag{2.3}$$

$$\int \frac{d\phi}{\sqrt{1+x\cos^{-1}\phi}} = \frac{1}{2}\cos^{-1}\phi^{m}\{\cos^{-1}\phi^{2}\}^{-m}\Big[(-\iota\cos^{-1}\phi)^{m}\ \Gamma(m+1,\iota\cos^{-1}\phi)+$$

$$+(\iota\cos^{-1}\phi)^{m}\Gamma(m+1,-\iota\cos^{-1}\phi)\Big]\ {}_1F_0\Big(\frac{1}{2};-;-x\Big) + \text{Constant}. \tag{2.4}$$

Derivation of (2.1):

$$\int \frac{d\phi}{\sqrt{1+x\sinh^{-1}\phi}} = \int (1+x\sinh^{-1}\phi)^{-\frac{1}{2}}d\phi = \int \{1-(-x\sinh^{-1}\phi)\}^{-\frac{1}{2}}d\phi$$

$$\int \sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(-x\sinh^{-1}\phi)^m}{m!}d\phi = \sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m(-x)^m}{m!}\int(\sinh^{-1}\phi)^m\ d\phi$$

$$= \frac{(-\sinh^{-1}\phi)^{-m}\sinh^{-1}\phi^{m}\ \Gamma(m+1,-\sinh^{-1}\phi)-\Gamma(m+1,\sinh^{-1}\phi)}{2}\ {}_1F_0\Big(\frac{1}{2};-;-x\Big).$$

Proceeding the same way one can established the other integrals.

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Cartesian product of fuzzy SU-ideals on SU-algebra

Utsanee Leerawat † and Rattana Sukklin ‡

Department of Mathematics Kasetsart University, Bangkok, Thailand
E-mail: fsciutl@ku.ac.th, rattana.nueng@gmail.com

Abstract In this paper, the notion of Cartesian product of fuzzy SU-ideals and Cartesian product of anti fuzzy SU-ideals on SU-algebra are introduced and some of their properties are investigated.

Keywords Cartesian product of fuzzy SU-ideal, Cartesian product of anti fuzzy SU-ideal.
2000 Mathematics Subject Classification: 03E72, 08A72

§1. Introduction and preliminaries

In 1965, Zadeh defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in continum, with each object being assigned a value between 0 and 1 by a membership function [1]. In 1991, Xi applied the concept of fuzzy set in BCK-algebras and defined fuzzy subalgebra on BCK-algebras [2]. Recently, a new algebraic structure was presented as SU-algebra and a concept of ideal in SU-algebra [3]. Sukklin and Leerawat introduced the concept of fuzzy SU-ideal [4] and anti fuzzy SU-ideal [5] in SU-algebra. The cartesian product of two fuzzy sets [6] was introduce by Bhattacharya and Mukherjee in 1985. In 2011, Samy M. Moutafa, et al. introduced the notion of Cartesian product of fuzzy KU-ideals [6] and Cartesian product of anti fuzzy KU-ideals [7] on KU-algebras. In this paper, we introduce the concept of cartesian product of fuzzy SU-ideals and cartesian product of anti fuzzy SU-ideals on SU-algebra. Moreover, we investigate some of their properties.

Definition 1.1. [3] A SU-algebra is a non-empty set X with a constant 0 and a binary operation "$*$"satisfying the following axioms:

(1) $((x*y)*(x*z))*(y*z)=0$,

(2) $x*0=x$,

(3) if $x*y=0$ imply $x=y$, for all $x,\ y,\ z\in X$.

From now on, a binary operation "$*$"will be denoted by juxtaposition.

Theorem 1.1. [3] Let X be a SU-algebra. Then the following results hold for all $x,\ y\in X$.

(1) $xx=0$, (2) $xy=yx$, (3) $0x=x$.

Theorem 1.2. [3] Let X be a SU-algebra. A nonempty subset I of X is called a SU-subalgebra of X if $xy\in I$ for all $x,\ y\in I$.

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[0]Corresponding author is Utsanee Leerawat

Definition 1.2. [3] Let X be a SU-algebra. A nonempty subset I of X is called a SU-ideal of X if it satisfies the following properties:

(1) $0 \in I$,

(2) if $(xy)z \in I$ and $y \in I$ imply $xz \in I$, for all $x,\ y,\ z \in X$.

Definition 1.3. [1] Let X be a set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$.

Definition 1.4. [4] Let X be a SU-algebra. A fuzzy set μ in X is called fuzzy SU-ideal of X if it satisfies the following conditions:

(F_1) $\mu(0) \geq \mu(x)$,

(F_2) $\mu(xz) \geq \min\{\mu((xy)z), \mu(y)\}$, for all $x,\ y,\ z \in X$.

Definition 1.5.[5] Let X be a SU-algebra. A fuzzy set μ^* in X is called anti fuzzy SU-ideal of X if it satisfies the following conditions:

(AF_1) $\mu^*(0) \leq \mu^*(x)$,

(AF_2) $\mu^*(xz) \leq \max\{\mu^*((xy)z), \mu^*(y)\}$, for all $x,\ y,\ z \in X$.

Example 1.1. Let $X = \{0, 1, 2, 3\}$ be a set in which operation $*$ is defined by the following:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is a SU-algebra [3].

Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(0) = 1, \mu(1) = 0.5$ and $\mu(2) = \mu(3) = 0$. Then μ is a fuzzy SU-ideal in X [4].

Define a fuzzy set $\mu^* : X \to [0, 1]$ by $\mu^*(0) = 0.3, \mu^*(1) = 0.6$ and $\mu^*(2) = \mu^*(3) = 0.7$. Then μ^* is an anti fuzzy SU-idealin X [5].

§2. Definition and properties

We first give the definition of cartesian product of fuzzy SU-ideals and cartesian product of anti fuzzy SU-ideals on SU-algebra and provide some its properties.

Definition 2.1. Let X be a SU-algebra and μ, β be a fuzzy SU-ideals of X. The Cartesian product $\mu \times \beta : X \times X \to [0, 1]$ is define by $(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$ for all $x,\ y \in X$.

Remark 2.1. If X be a SU-algebra, then $\underbrace{X \times X \times \cdots \times X}_{n}$ is a SU-algebra, where n is positive intergers.

Theorem 2.1. Let X be a SU-algebra and $\mu,\ \beta$ be a fuzzy sets of X. If μ and β are a fuzzy SU-ideals of X, then $\mu \times \beta$ is a fuzzy SU-ideal of $X \times X$.

Proof. Let $\mu,\ \beta$ be a fuzzy SU-ideals of X.

(i) Let $(x_1, x_2) \in X \times X$, we have

$$\begin{aligned}(\mu \times \beta)(0,0) \quad &= \min\{\mu(0), \beta(0)\} \\ &\geq \min\{\mu(x_1), \beta(x_2)\} \\ &= (\mu \times \beta)(x_1, x_2).\end{aligned}$$

Thus $(\mu \times \beta)(0,0) \geq (\mu \times \beta)(x_1, x_2)$ for all $(x_1, x_2) \in X \times X$.

(ii) Let $(x_1, x_2),\ (y_1, y_2),\ (z_1, z_2) \in X \times X$, we have

$$\begin{aligned}&(\mu \times \beta)((x_1, x_2)(z_1, z_2)) = (\mu \times \beta)(x_1 z_1, x_2 z_2) \\ &= \min\{\mu(x_1 z_1), \beta(x_2 z_2)\} \geq \min\{\min\{\mu((x_1 y_1) z_1), \mu(y_1)\}, \min\{\beta((x_2 y_2) z_2), \beta(y_2)\}\} \\ &= \min\{\min\{\mu((x_1 y_1) z_1), \beta((x_2 y_2) z_2)\}, \min\{\mu(y_1), \beta(y_2)\}\} \\ &= \min\{(\mu \times \beta)((x_1 y_1) z_1, (x_2 y_2) z_2), (\mu \times \beta)(y_1, y_2)\}.\end{aligned}$$

Thus $(\mu \times \beta)((x_1, x_2)(z_1, z_2)) \geq \min\{(\mu \times \beta)(((x_1, x_2)(y_1, y_2))(z_1, z_2)), (\mu \times \beta)(y_1, y_2)\}$.
Therefore $\mu \times \beta$ is a fuzzy SU-ideal of $X \times X$.

Corollary 2.1 Let X be a SU-algebra and $\mu_1,\ \mu_2,\ \cdots,\ \mu_n$ be a fuzzy sets of X. If $\mu_1,\ \mu_2,\ \cdots,\ \mu_n$ are a fuzzy SU-ideals of X, then $\mu_1 \times \mu_2 \times \cdots \times \mu_n$ is a fuzzy SU-ideal of $\underbrace{X \times X \times \cdots \times X}_{n}$, where n is positive intergers.

Theorem 2.2. Let X be a SU-algebra and μ, β be a fuzzy sets of X such that $\mu \times \beta$ is a fuzzy SU-ideal of $X \times X$, then

(i) Either $\mu(0) \geq \mu(x)$ or $\beta(0) \geq \beta(x)$ for all $x \in X$.

(ii) If $\mu(0) \geq \mu(x)$ for all $x \in X$, then either $\beta(0) \geq \mu(x)$ or $\beta(0) \geq \beta(x)$.

(iii) If $\beta(0) \geq \beta(x)$ for all $x \in X$, then either $\mu(0) \geq \beta(x)$ or $\mu(0) \geq \mu(x)$.

(iv) Either μ or β is a fuzzy SU-ideal of X.

Proof. Let $\mu \times \beta$ is a fuzzy SU-ideal of $X \times X$ and $x,\ y,\ z \in X$.

(i) Assume $\mu(0) \geq \mu(x)$ or $\beta(0) \geq \beta(x)$ is not true, there exists $(a, b) \in X \times X$ such that $\mu(0) < \mu(a)$ and $\beta(0) < \beta(b)$. We have

$$\begin{aligned}(\mu \times \beta)(0,0) \quad &= \min\{\mu(0), \beta(0)\} \\ &< \min\{\mu(a), \beta(b)\} \\ &= (\mu \times \beta)(a, b).\end{aligned}$$

Hence $(\mu \times \beta)(0,0) < (\mu \times \beta)(a, b)$, which is contradiction. Therefore either $\mu(0) \geq \mu(x)$ or $\beta(0) \geq \beta(x)$ for all $x \in X$.

(ii) Let $\mu(0) \geq \mu(x)$ for all $x \in X$. Assume $\beta(0) \geq \mu(x)$ or $\beta(0) \geq \beta(x)$ is not true, there exists $(c, d) \in X \times X$ such that $\beta(0) < \mu(c)$ and $\beta(0) < \beta(d)$. We have $(\mu \times \beta)(0,0) = \min\{\mu(0), \beta(0)\} = \beta(0)$. So $(\mu \times \beta)(c, d) = \min\{\mu(c), \beta(d)\} > \beta(0) = (\mu \times \beta)(0,0)$. Hence $(\mu \times \beta)(c, d) > (\mu \times \beta)(0,0)$, which is contradiction. Therefore if $\mu(0) \geq \mu(x)$ for all $x \in X$, then either $\beta(0) \geq \mu(x)$ or $\beta(0) \geq \beta(x)$.

(iii) This proof is quite similar to (ii).

(iv) (1) Let $\mu(0) \geq \mu(x)$ for all $x \in X$.

By Theorem 3.4(ii), we have either $\beta(0) \geq \mu(x)$ or $\beta(0) \geq \beta(x)$.

If $\beta(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(x) = \min\{\mu(x), \beta(0)\}$. Hence $\mu(x) = (\mu \times \beta)(x, 0)$. So

$$\begin{aligned} \mu(xz) &= \min\{\mu(xz), \beta(0)\} \\ &= (\mu \times \beta)(xz, 0) \\ &\geq \min\{(\mu \times \beta)((xy)z), 0), (\mu \times \beta)(y, 0)\} \\ &= \min\{\mu((xy)z), \mu(y)\}. \end{aligned}$$

Thus μ is a fuzzy SU-ideal of X.

If $\beta(0) \geq \beta(x)$ for all $x \in X$, then $\beta(0) < \mu(x)$ for all $x \in X$. We have $\beta(x) = \min\{\mu(0), \beta(x)\}$. Hence $\beta(x) = (\mu \times \beta)(0, x)$. So

$$\begin{aligned} \beta(xz) &= \min\{\mu(0), \beta(xz)\} \\ &= (\mu \times \beta)(0, xz) \\ &\geq \min\{(\mu \times \beta)(0, (xy)z)), (\mu \times \beta)(0, y)\} \\ &= \min\{\beta((xy)z), \beta(y)\}. \end{aligned}$$

Thus β is a fuzzy SU-ideal of X.

(2) In case $\beta(0) \geq \beta(x)$ for all $x \in X$, a proof is similar to (1).

Definition 2.2. Let X be a SU-algebra and μ^*, β^* be an anti fuzzy SU-ideals of X. The Cartesian product $\mu^* \times \beta^* : X \times X \to [0, 1]$ is define by $(\mu^* \times \beta^*)(x, y) = \max\{\mu^*(x), \beta^*(y)\}$ for all $x,\ y \in X$.

Theorem 2.3. Let X be a SU-algebra and $\mu^*,\ \beta^*$ be a fuzzy sets of X. If μ^* and β^* are an anti fuzzy SU-ideals of X, then $\mu^* \times \beta^*$ is an anti fuzzy SU-ideal of $X \times X$.

Proof. Let $\mu^*,\ \beta^*$ be an anti fuzzy SU-ideals of X.

(i) Let $(x_1, x_2) \in X \times X$, we have

$$\begin{aligned} (\mu^* \times \beta^*)(0, 0) &= \max\{\mu^*(0), \beta^*(0)\} \\ &\leq \max\{\mu^*(x_1), \beta^*(x_2)\} \\ &= (\mu^* \times \beta^*)(x_1, x_2). \end{aligned}$$

Thus $(\mu^* \times \beta^*)(0, 0) \leq (\mu^* \times \beta^*)(x_1, x_2)$ for all $(x_1, x_2) \in X \times X$.

(ii) Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have

$$\begin{aligned} (\mu^* \times \beta^*)((x_1, x_2)(z_1, z_2)) &= (\mu^* \times \beta^*)(x_1z_1, x_2z_2) \\ &= \max\{\mu^*(x_1z_1), \beta^*(x_2z_2)\} \\ &\leq \max\{\max\{\mu^*((x_1y_1)z_1), \mu^*(y_1)\}, \max\{\beta^*((x_2y_2)z_2), \beta^*(y_2)\}\} \\ &= \max\{\max\{\mu^*((x_1y_1)z_1), \beta^*((x_2y_2)z_2)\}, \max\{\mu^*(y_1), \beta^*(y_2)\}\} \\ &= \max\{(\mu^* \times \beta^*)((x_1y_1)z_1, (x_2y_2)z_2), (\mu^* \times \beta^*)(y_1, y_2)\}. \end{aligned}$$

Thus $(\mu^*\times\beta^*)((x_1, x_2)(z_1, z_2)) \leq \max\{(\mu^* \times \beta^*)(((x_1, x_2)(y_1, y_2))(z_1, z_2)), (\mu^* \times \beta^*)(y_1, y_2)\}$.

Therefore $\mu^* \times \beta^*$ is an anti fuzzy SU-ideal of $X \times X$.

Corollary 2.2. Let X be a SU-algebra and $\mu_1^*,\ \mu_2^*,\ \cdots,\ \mu_n^*$ be a fuzzy sets of X. If $\mu_1^*,\ \mu_2^*,\ \cdots,\ \mu_n^*$ are an anti fuzzy SU-ideals of X, then $\mu_1^* \times \mu_2^* \times \cdots \times \mu_n^*$ is an anti fuzzy SU-ideal of $\underbrace{X \times X \times \cdots \times X}_{n}$, where n is positive intergers.

Theorem 2.4. Let X be a SU-algebra and μ^*, β^* be a fuzzy sets of X such that $\mu^* \times \beta^*$ is an anti fuzzy SU-ideal of $X \times X$, then

(i) Either $\mu^*(0) \leq \mu^*(x)$ or $\beta^*(0) \leq \beta^*(x)$ for all $x \in X$.

(ii) If $\mu^*(0) \leq \mu^*(x)$ for all $x \in X$, then either $\beta^*(0) \leq \mu^*(x)$ or $\beta^*(0) \leq \beta^*(x)$.

(iii) If $\beta^*(0) \leq \beta^*(x)$ for all $x \in X$, then either $\mu^*(0) \leq \beta^*(x)$ or $\mu^*(0) \leq \mu^*(x)$.

(iv) Either μ^* or β^* is an anti fuzzy SU-ideal of X.

Proof. Let $\mu^* \times \beta^*$ is an anti fuzzy SU-ideal of $X \times X$ and $x,\ y,\ z \in X$.

(i) Assume $\mu^*(0) \leq \mu^*(x)$ or $\beta^*(0) \leq \beta^*(x)$ is not true, there exists $(a, b) \in X \times X$ such that $\mu^*(0) > \mu^*(a)$ and $\beta^*(0) > \beta^*(b)$. We have $(\mu^* \times \beta^*)(0, 0) = \max\{\mu^*(0), \beta^*(0)\} > \max\{\mu^*(a), \beta^*(b)\} = (\mu^* \times \beta^*)(a, b)$. Hence $(\mu^* \times \beta^*)(0, 0) > (\mu^* \times \beta^*)(a, b)$, which is contradiction. Therefore either $\mu^*(0) \leq \mu^*(x)$ or $\beta^*(0) \leq \beta^*(x)$ for all $x \in X$.

(ii) Let $\mu^*(0) \leq \mu^*(x)$ for all $x \in X$. Assume $\beta^*(0) \leq \mu^*(x)$ or $\beta^*(0) \leq \beta^*(x)$ is not true, there exists $(c, d) \in X \times X$ such that $\beta^*(0) > \mu^*(c)$ and $\beta^*(0) > \beta^*(d)$. We have

$$\begin{aligned} (\mu^* \times \beta^*)(0, 0) \qquad &= \max\{\mu^*(0), \beta^*(0)\} \\ &= \beta^*(0). \end{aligned}$$

So

$$\begin{aligned} (\mu^* \times \beta^*)(c, d) \qquad &= \max\{\mu^*(c), \beta^*(d)\} < \beta^*(0) \\ &= (\mu^* \times \beta^*)(0, 0). \end{aligned}$$

Hence $(\mu^* \times \beta^*)(c, d) < (\mu^* \times \beta^*)(0, 0)$, which is contradiction. Therefore if $\mu^*(0) \leq \mu^*(x)$ for all $x \in X$, then either $\beta^*(0) \leq \mu^*(x)$ or $\beta^*(0) \leq \beta^*(x)$.

(iii) This proof is quite similar to (ii).

(iv) (1) Let $\mu^*(0) \leq \mu^*(x)$ for all $x \in X$.

By Theorem 4.4(ii), we have either $\beta^*(0) \leq \mu^*(x)$ or $\beta^*(0) \leq \beta^*(x)$.

If $\beta^*(0) \leq \mu^*(x)$ for all $x \in X$, we have $\mu^*(x) = \max\{\mu^*(x), \beta^*(0)\}$.

Hence $\mu^*(x) = (\mu^* \times \beta^*)(x, 0)$.

So

$$\begin{aligned} \mu^*(xz) \qquad &= \max\{\mu^*(xz), \beta^*(0)\} \\ &= (\mu^* \times \beta^*)(xz, 0) \\ &\leq \max\{(\mu^* \times \beta^*)((xy)z), 0), (\mu^* \times \beta^*)(y, 0)\} \\ &= \max\{\mu^*((xy)z), \mu(y)^*\}. \end{aligned}$$

Thus μ^* is an anti fuzzy SU-ideal of X.

If $\beta^*(0) \leq \beta^*(x)$ for all $x \in X$, then $\beta^*(0) > \mu^*(x)$ for all $x \in X$.

We have $\beta^*(x) = \max\{\mu^*(0), \beta^*(x)\}$. Hence $\beta^*(x) = (\mu^* \times \beta^*)(0, x)$.

So

$$\begin{aligned}\beta^*(xz) = \max\{\mu^*(0), \beta^*(xz)\} &= (\mu^* \times \beta^*)(0, xz)\\ &\leq \max\{(\mu^* \times \beta^*)(0, (xy)z)), (\mu^* \times \beta^*)(0, y)\}\\ &= \max\{\beta^*((xy)z), \beta^*(y)\}.\end{aligned}$$

Thus β^* is an anti fuzzy SU-ideal of X.

(2) In case $\beta^*(0) \leq \beta^*(x)$ for all $x \in X$, a proof is similar to (1).

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